9.23 Reflecting random walk on the line. Consider the points 1, 2, 3, 4 to be marked on a straight line. Let $X_n$ be a Markov chain that moves to the right with probability $\frac{2}{3}$ and to the left with probability $\frac{1}{3}$, but subject this time to the rule that if $X_n$ tries to go to the left from 1 or to the right from 4 it stays put. Find (a) the transition probability for the chain, and the (b) the limiting amount of time the chain spends at each site.

(a)

\[
\begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} \\
0 & 0 & \frac{1}{3} & \frac{2}{3}
\end{pmatrix}
\]

(b) The limiting distribution of time spent in each state is $\frac{1}{15} (1, 4, 8)$.

In words, if he last shaved $k$ days ago, he will not shave with probability $\frac{1}{k+1}$. However, when has not shaved for 4 days his wife orders him to shave, and he does so with probability 1. (a) What is the long-run fraction of time Mickey shaves? (b) Does the stationary distribution for this chain satisfy the detailed balance condition?

(a) The limiting distribution of time spent in each state is $\frac{1}{41} (24, 12, 4, 1)$. It follows that the long run fraction of days that Mickey shaves is $\frac{24}{41}$.

(b) No. In particular

$0 = \pi(1)p(1,3) \neq \pi(3)p(3,1) = \frac{8}{123}$.

9.26 Let $X_n$ be the number of days since Miceky Markov last shaved, calculated at 7:30AM when he is trying to decide if he wants to shave today. Suppose that $X_n$ is a Markov chain with transition matrix

\[
\begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

In words, if he last shaved $k$ days ago, he will not shave with probability $\frac{1}{k+1}$. However, when has not shaved for 4 days his wife orders him to shave, and he does so with probability 1. (a) What is the long-run fraction of time Mickey shaves? (b) Does the stationary distribution for this chain satisfy the detailed balance condition?

(b) There are several ways to do this problem. The following way is more intuitive.
The transition probability of the Markov chain is
\[
\begin{pmatrix}
.2 & .7 & .1 & 0 \\
.05 & .1 & .05 & .8 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
By simple calculation we know \( \pi_1 = \pi_2 = 0 \) while \( \pi_3 \) and \( \pi_4 \) are undetermined. From the problem we know a part will always start from step 1 and end at either step 3 or 4. In other words we only need to know the value of \( p^n(1, 3) \) for \( n \to \infty \). Since \( \pi_1 = \pi_2 = 0 \), we have \( p^n(i, 1) = p^n(i, 2) = 0 \) for \( i = 1, 2, 3, 4 \), and \( n \to \infty \). By the property of Markov chain
\[
p^n(k, 3) = \sum_{j=1}^{4} p(k, j)p^{n-1}(j, 3)
\]
Let \( a_{ij} = \lim_{n \to \infty} p^n(i, j) \) as a brief notation. If \( a_{ij} \) exist and unique, then for \( n \to \infty \), the above equation can be written as
\[
a_{13} = \sum_{j=1}^{4} p(1, j)a_{j3} = .2a_{13} + .7a_{23} + .1a_{33} + (0)a_{43}
\]
\[
a_{23} = \sum_{j=1}^{4} p(2, j)a_{j3} = .05a_{13} + .1a_{23} + .05a_{33} + .8a_{43}
\]
\[
\vdots
\]
It is obvious that \( a_{33} = a_{44} = 1 \), \( a_{34} = a_{43} = 0 \). So we have
\[
a_{13} = .2a_{13} + .7a_{23} + .1
\]
\[
a_{23} = .05a_{13} + .1a_{23} + .05.
\]
By solving the system of equations \( a_{13} = .1825, a_{23} = .0657 \) and the probability a part is scrapped is .1825.

9.9(b) Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

(a) The transition probability of the Markov chain is
\[
\begin{pmatrix}
D & H & J & M & S & T \\
D & 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\
H & 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\
J & 0 & 0 & 0 & 0 & 0 & 0 \\
M & 0 & 0 & 0 & 1 & 0 & 0 \\
S & 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\
T & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
and we need to find the value of \( a_1 \), where \( a_i = P_i(\text{Mark ends up with the ball}) \). Conditioning on the first step we obtain:
\[
a_k = \sum_{j=1}^{6} p(k, j)a_j \text{ for } k = 1, \ldots, 6.
\]

2
After simplification we have

\[ a_1 = \frac{1}{4}a_2 + \frac{1}{4}a_5 + \frac{1}{4} \\
 a_2 = \frac{1}{4}a_1 + \frac{1}{4}a_5 \\
 a_5 = \frac{1}{4}a_1 + \frac{1}{4}a_2 + \frac{1}{4} \]

and \( a_1 = .4, a_2 = .2, a_5 = .4 \). So the probability Mark will end up with the ball is .4.

**9.34 Coupon collector’s problem.** We are interested now in the time it takes to collect a set of \( N \) baseball cards. Let \( T_k \) be the number of cards we have to buy before we have \( k \) that are distinct. Clearly, \( T_1 = 1 \). A little more thought reveals that if each time we get a card chosen at random from all \( N \) possibilities, then for \( k \geq 1 \), \( T_{k+1} - T_k \) has a geometric distribution with success probability \( (N - k)/N \). Use this to show that the mean time to collect a set of \( N \) baseball cards is \( \sim N \log N \), while the variance is \( \sim N^2 \sum_{k=1}^{\infty} 1/k^2 \).

Since \( T_{k+1} - T_k \) has a geometrix distribution with probability \( (N - k)/N \),

\[ E(T_{k+1} - T_k) = \left( \frac{N - k}{N} \right)^{-1} = \frac{N}{N - k} \]

So

\[ E(T_{k+1}) = E(T_k) + E(T_{k+1} - T_k) = E(T_k) + \frac{N}{N - k} \]

\[ E(T_N) = E(T_{N-1}) + \frac{N}{N - (N-1)} = E(T_{N-2}) + \frac{N}{N - (N-2)} + \frac{N}{N - (N-1)} = \cdots \]

\[ = E(T_1) + \sum_{j=1}^{N-1} \frac{N}{N - j} = 1 + N \sum_{j=1}^{N-1} \frac{1}{j} \]

\[ \sim N \log N. \]

Since \( \{T_k\} \) are independent to each other, \( Var(T_{k+1}) = Var(T_{k+1} - T_k) + Var(T_k) \).

\[ Var(T_{k+1} - T_k) = \frac{1 - \frac{N - k}{N}}{(N - k)^2} = \frac{Nk}{(N - k)^2}. \]

So

\[ Var(T_{k+1}) = Var(T_k) + \frac{Nk}{(N - k)^2} \]

\[ Var(T_N) = Var(T_{N-1}) + \frac{N(N-1)}{(N - (N-1))^2} = \cdots \]

\[ = Var(T_1) + \sum_{j=1}^{N-1} \frac{Nj}{(N - j)^2} \]

\[ = \sum_{j=1}^{N-1} \frac{N(N - j)}{j^2} = \sum_{j=1}^{N-1} \left( \frac{N^2}{j^2} - \frac{N}{j} \right) \]

\[ \sim N^2 \sum_{k=1}^{\infty} \frac{1}{k^2}. \]
A criminal named Xavier and a policeman named Yacov move between three possible hideouts according to Markov chains $X_n$ and $Y_n$ with transition probabilities:

$$p_{Xavier} = \begin{pmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{pmatrix} \quad \text{and} \quad p_{Yakov} = \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{pmatrix}$$

At time $T = \min\{n : X_n = Y_n\}$ the game is over and the criminal is caught. (a) Suppose $X_0 = i$ and $Y_0 = j \neq i$. Find the expect value of $T$. (b) Suppose that the two players generalize their strategies to

$$p_{Xavier} = \begin{pmatrix} 1 - 2p & p & p \\ p & 1 - 2p & p \\ p & p & 1 - 2p \end{pmatrix} \quad \text{and} \quad p_{Yakov} = \begin{pmatrix} 1 - 2q & q & q \\ q & 1 - 2q & q \\ q & q & 1 - 2q \end{pmatrix}$$

If Yakov uses $q = .5$ as he did in part (a), what values of $p$ should Xavier choose to maximize the expected time to get caught? Answer the last question again for $q = 1/3$.

(a) Observe that the terms in the transition matrices for $X$ and $Y$ are each constant along the diagonal as well off-diagonal. Let 1,2,3 be the three states where $X$ and $Y$ stay. Without loss of generality, suppose $X$ is at 1 and $Y$ is at 2 at time $k$. At time $k+1$, the probability of getting caught or not is shown as

<table>
<thead>
<tr>
<th></th>
<th>caught</th>
<th>not caught</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X=1, Y=1$</td>
<td>.3</td>
<td></td>
</tr>
<tr>
<td>$X=1, Y=3$</td>
<td>.3</td>
<td></td>
</tr>
<tr>
<td>$X=2, Y=1$</td>
<td>.1</td>
<td></td>
</tr>
<tr>
<td>$X=2, Y=3$</td>
<td>.1</td>
<td></td>
</tr>
<tr>
<td>$X=3, Y=1$</td>
<td>.1</td>
<td></td>
</tr>
<tr>
<td>$X=3, Y=3$</td>
<td>.1</td>
<td></td>
</tr>
</tbody>
</table>

We can generalize a new Markov chain model with 2 states “caught” and “not caught”. The transition probability is

<table>
<thead>
<tr>
<th></th>
<th>caught</th>
<th>not caught</th>
</tr>
</thead>
<tbody>
<tr>
<td>caught</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>not caught</td>
<td>.4</td>
<td>.6</td>
</tr>
</tbody>
</table>

The expect caught time is therefore

$$ET = \sum_{k=1}^{\infty} P(N \geq k) = \sum_{k=0}^{\infty} (.6)^k = \frac{1}{1 - .6} = 2.5$$

(b) Since the 3 states are still symmetric, we can generalize the table (suppose $X = 1$ and $Y = 2$ at time $k$):
Counting the probability:

\[(1 - 2p)q + p(1 - 2q) = p + q - 3pq\]

\[(1 - 2p)(1 - 2q) + (1 - 2p)q + pq + pq + pq + p(1 - 2q) = 1 - (p + q - 3pq).\]

The transition matrix is

\[
\begin{array}{cc}
\text{caught} & \text{not caught} \\
X=1, Y=1 & (1-2p)q \\
X=1, Y=2 & (1-2p)(1-2q) \\
X=1, Y=3 & (1-2p)q \\
X=2, Y=1 & pq \\
X=2, Y=2 & p(1-2q) \\
X=2, Y=3 & pq \\
X=3, Y=1 & pq \\
X=3, Y=2 & p(1-2q) \\
X=3, Y=3 & pq \\
\end{array}
\]

The expect caught time would be

\[ET = \frac{(1 + p)/2}{1 - (1 + p)/2} = \frac{1 + p}{1 - p}.\]

To maximize the time to be caught, we look at the first derivative of \(ET\):

\[\frac{d}{dp} \left( \frac{1 + p}{1 - p} \right) = \frac{(1 - p) + (1 + p)}{(1 - p)^2} = \frac{2}{(1 - p)^2} > 0.\]

The maximum value \(p\) can have is .5, so \(\max \{ET\} = 1.5/.5 = 3.\)

When \(q = .5\), \(1 - (p + q - 3pq) = 2/3\),

\[ET = \frac{2/3}{1 - 2/3} = 2.\]

9.41 A warehouse has a capacity to hold four items. If the warehouse is neither full nor empty, the number of items in the warehouse changes whenever a new item is produced or an item is sold. Suppose that (no matter when we look) the probability that the next event is “a new item is produced” is \(\frac{2}{3}\) and that the new event is a “sale” is \(\frac{1}{3}\). If there is currently one item in the warehouse, what is the probability that the warehouse will become full before it becomes empty.

We create a Markov chain with absorbing states at empty and full. That is,

\[
\begin{pmatrix}
\text{empty} & 1 & 2 & 3 & \text{full} \\
\text{empty} & 1 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\
2 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\
3 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\
\text{full} & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Let \( P(i) \) be the probability that at state \( i \) the warehouse becomes full before becoming empty. Then using that \( P(0) = 0 \), and \( P(4) = 1 \), it follows that \( P(1) = \frac{2}{3} P(2), P(2) = \frac{1}{3} P(1) + \frac{2}{3} P(3) \), and \( P(3) = \frac{1}{3} P(2) + \frac{2}{3} \).

Solving this linear system gives that \( A\vec{P} = \vec{b} \) where
\[
A = \begin{pmatrix} -1 & \frac{2}{3} & 0 \\ \frac{1}{3} & -1 & \frac{2}{3} \\ 0 & \frac{1}{3} & -1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} P(1) \\ P(2) \\ P(3) \end{pmatrix}.
\]

Solving the system gives that \( \vec{b} = \frac{1}{17}(4, 6, 7)' \). In particular \( P(1) = \frac{4}{17} \).

9.46 General birth and death chains. The state space is \( \{0, 1, 2, \ldots\} \) and the transition probability has
\[
p(x, x + 1) = p_x \\
p(x, x - 1) = q_x \quad \text{for } x > 0 \\
p(x, x) = r_x \quad \text{for } x \geq 0
\]
while the other \( p(x, y) = 0 \). Let \( V_y = \min\{n \geq 0 : X_n = y\} \) be the time of the first visit to \( y \) and let \( h_N(x) = P_x(V_N < V_0) \).

By considering what happens on the first step, we can write
\[
h_N(x) = p_x h_N(x + 1) + r_x h_N(x) + q_x h_N(x - 1).
\]

Set \( h_N(1) = c_N \) and solve this equation to conclude that 0 is recurrent if and only if \( \sum_{y=1}^{\infty} \prod_{x=1}^{y-1} q_x/p_x = \infty \) where by convention \( \prod_{x=1}^{0} = 1 \).

First, \( h_N(0) = P_0(V_N < V_0) = 0 \) from the definition.

Second, we find the solution of \( h_N(x) \). From the one-step equation,
\[
h_N(x + 1) = \frac{1}{p_x} [(1 - r_x) h_N(x) - q_x h_N(x - 1)] \\
= \frac{1}{p_x} [(p_x + q_x) h_N(x) - q_x h_N(x - 1)] \\
= h_N(x) + \frac{q_x}{p_x} (h_N(x) - h_N(x - 1))
\]

We will prove
\[
h_N(x) = c_N \sum_{i=1}^{x} \prod_{j=1}^{i-1} \frac{q_j}{p_j}
\]
by induction.

The equality holds when \( x = 1 \). Suppose it holds when \( x = t \), then when \( x = t + 1 \),
\[
h_N(t + 1) = h_N(t) + \frac{q_t}{p_t} (h_N(t) - h_N(t - 1)) \\
= c_N \sum_{i=1}^{t} \prod_{j=1}^{i-1} \frac{q_j}{p_j} + \frac{q_t}{p_t} c_N \prod_{j=1}^{t-1} \frac{q_j}{p_j} \\
= c_N \left( \sum_{i=1}^{t} \prod_{j=1}^{i-1} \frac{q_j}{p_j} + \frac{t}{p_t} \prod_{j=1}^{t} \frac{q_j}{p_j} \right) \\
= c_N \sum_{i=1}^{t+1} \prod_{j=1}^{i-1} \frac{q_j}{p_j}
\]
Thus $h_N(x) = c_N \sum_{i=1}^{x} \prod_{j=1}^{i-1} \frac{q_j}{p_j}$ is the solution.

Third, we find the form of $c_N$. Since $h_N(N) = P_N(V_N < V_0) = 1$ by definition,

$$h_N(N) = c_N \sum_{i=1}^{N} \prod_{j=1}^{i-1} \frac{q_j}{p_j} = 1$$

$$c_N = \left( \sum_{i=1}^{N} \prod_{j=1}^{i-1} \frac{q_j}{p_j} \right)^{-1}.$$ 

$c_N = h_N(1)$ and is the probability that we start at 1 and visit $N$ before visiting 0. Define

$$c_\infty = \lim_{N \to \infty} c_N$$

$$= \left( \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \frac{q_j}{p_j} \right)^{-1}$$

to be the probability that we start at 1 and visit 1,2,3,... before visiting 0.

Fourth, we make the proof.

If $\sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \frac{q_j}{p_j} = \infty$, $c_\infty = 0$. Then it is certain we will visit 0. And since we can only visit 0 from 0 or 1, we will visit 0 infinitely often, so 0 is recurrent. If $\sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \frac{q_j}{p_j} > 0$, $c_\infty > 0$. We will visit every state before visiting 0, which means we never visit 0 so 0 is transient.

9.48 Consider the Markov chain with state space \{0,1,2,...\} and transition probability

$$p(m, m+1) = \frac{1}{2} \left( 1 - \frac{1}{m + 2} \right) \text{ for } m \geq 0$$

$$p(m, m-1) = \frac{1}{2} \left( 1 + \frac{1}{m + 2} \right) \text{ for } m \geq 1$$

and $p(0,0) = 1 - p(0,1) = 3/4$. Find the stationary distribution $\pi$.

Since this is a birth-death chain, we exploit the detailed-balanced condition:

$$v_{m+1} = \frac{p(m, m+1)}{p(m+1, m)} \cdot v_m = \frac{(m+1)(m+3)}{(m+2)(m+4)} \cdot v_m.$$ 

Iterating the recursion leads to

$$v_1 = v_0 \cdot \frac{1 \cdot 3}{2 \cdot 4};$$

$$v_2 = v_0 \cdot \frac{1 \cdot 3 \cdot 2 \cdot 4}{2 \cdot 4 \cdot 3 \cdot 5};$$

$$v_3 = v_0 \cdot \frac{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 3 \cdot 5 \cdot 4 \cdot 6};$$

and so on, which simplifies to

$$v_1 = \frac{3v_0}{2 \cdot 4};$$

$$v_2 = \frac{3v_0}{3 \cdot 5};$$

$$v_3 = \frac{3v_0}{4 \cdot 6};$$
from which the following pattern is apparent (you can easily check that this solves the detailed-balanced condition):

\[ v_m = \frac{3v_0}{(m+1) \cdot (m+3)} \]

Now to determine the value of \( v_0 \) by imposing that \( \sum_{n=0}^{\infty} \pi_n = 1 \):

\[
1 = \sum_{n=0}^{\infty} \frac{3v_0}{(n+1)(n+3)} = \frac{3v_0}{2} \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+3} \right)
\]

\[
= \frac{3v_0}{2} \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{3v_0}{2} \left(1 + \frac{1}{2}\right) = \frac{9}{4} v_0.
\]

so \( v_0 = 4/9 \) and

\[
\pi_n = \frac{3}{4(n+1)(n+3)}, \text{ for all } n \geq 0.
\]

9.53 Consider a branching process as defined in Example 7.2, in which each family has a number of children that follows a shifted geometric distribution: \( p_k = p(1-p)^k \) for \( k \geq 0 \), which counts the number of failures before the first success when success has probability \( p \). Compute the probability that starting from one individual the chain will be absorbed at 0. Let \( p_e \) be the extinction probability starting at one individual. It follows that

\[
p_e = \sum_{k=1}^{\infty} p(1-p)^k (p_e)^k
\]

\[
= \frac{1}{p(1-p)} \sum_{k=1}^{\infty} ((1-p)(p_e))^{k-1}
\]

\[
= \frac{1}{p(1-p)} = \frac{1}{p_e(1-p)}.
\]

It follows that \( p_e = p \), and hence the probability that the chain will be absorbed at 0, starting with one individual is \( p \).

7.1 Suppose that the time to repair a machine is exponentially distributed random variable with mean 2. (a) What is the probability the repair takes more than 2 hours. (b) What is the probability that the repair takes more than 5 hours given that it takes more than 3 hours.

(a) Since the mean is 2, the rate is 1/2. \( P(T > 2) = e^{-1/2} = e^{-1} \).

(b) \( P(T > 5 | T > 3) = P(T > 2) = e^{-1} \).

7.2 The lifetime of radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?

The rate is 1/5. \( P(T > 10 | T > 7) = P(T > 3) = e^{-3/5} \).