CS-211 - Sp04
Week 6 - Solutions

(Section 4.5 p166-167)

8. $S_n = S_{n-1} + 2S_{n-2}$, $n \geq 2$, $S_0 = 3$, $S_1 = -3$.

$S_n - S_{n-1} - 2S_{n-2} = 0$ is a homogeneous, linear recurrence relation (with constant coefficients). Its characteristic eqn:

$\alpha^2 - \alpha - 2 = 0$ has roots $2, -1$ $\left[(\alpha-2)(\alpha+1)=0\right]$

so the general solution will be of the form:

$S_n = c_1 2^n + c_2(-1)^n$. Solving for a exact sol.

so $S_0 = 3 = c_1 + c_2$, and
$S_1 = -3 = 2c_1 - c_2$, so $c_1 = 0$, $c_2 = 3$ and

our <u>exact</u> solution will be.

$S_n = 3(-1)^n$ for all $n \geq 0$.

10 A) 3, 4, 7, 11, 18

B) $S_n = S_{n-1} + S_{n-2}$, $n \geq 2$, $S_0 = 2$, $S_1 = 1$, yields the characteristic equation $\alpha^2 - \alpha - 1 = 0$. The same char. eqn as the Fibonacci sequence. The roots are

$r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. So our general solution will be $S_n = c_1 r_1^n + c_2 r_2^n$. Solving for c_1 and c_2 we find $c_1 = c_2 = 1$, so our exact solution will be.

$$S_n = r_1^n + r_2^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \forall n \geq 0.$$

(Section 3.1, p. 99-100)

4. A $\{(0,5), (1,4), (2,3), (3,2), (4,1), (5,0)\}$

B. $\{(0,2), (1,2), (2,2), (2,1), (2,0)\}$

C. $(2,2)$, $(73,2)$, $(2,17)$, are some of the pairs.

6. R is reflexive. For any $x \in \mathbb{Z}$, $x^3 - x^3 = 0 \equiv 0 \bmod 5$, so xRx.

R is symmetric. Suppose xRy, then $x^3 - y^3 \equiv 0 \bmod 5$. But, then $x^3 - y^3 = k \cdot 5$, for some $k \in \mathbb{Z}$, and this implies $y^3 - x^3 = -k \cdot 5$. Therefore $y^3 - x^3 \equiv 0 \bmod 5$, so yRx.

R is transitive. Suppose xRy and yRz, then $x^3 - y^3 = k_1 \cdot 5$ for some $k_1 \in \mathbb{Z}$, and $y^3 - z^3 = k_2 \cdot 5$, for some $k_2 \in \mathbb{Z}$. This implies (with a little algebra) that

$x^3 - z^3 = (k_1 + k_2) \cdot 5$, so $x^3 - z^3 \equiv 0 \bmod 5$ and xRz.

R is not Antisymmetric, since $5R0$ and $0R5$, but $0 \neq 5$.

10. A) the relation $<$ on $\mathbb{N}$

B) The relation $\neq$ on $\mathbb{N}$.

12. Prove $\Rightarrow$ (Show: R is transitive $\Rightarrow R^{\leftarrow}$ is transitive)

Suppose $xR^{\leftarrow}y$ and $yR^{\leftarrow}z$. Then by def. of converse zRy and yRx. Since R is transitive, we know zRx. But, then $xR^{\leftarrow}z$, so $R^{\leftarrow}$ is transitive.

$\Leftarrow$ Handled in similar fashion.

14. A) Done in class

B)

C) NO. Counterexample: Let $S = \{a, b, c\}$, $R_1 = \{(a,b)\}$, $R_2 = \{(b,c)\}$. R_1 and R_2 are transitive (trivially), but $R_1 \cup R_2 = \{(a,b), (b,c)\}$ is not transitive.

16. Done in class.

(Section 3.4 118-119)

2. A) $[L]$ is the family of all lines parallel to L, including L itself.

(B), (c), (d), (e) have no equivalence classes

f) $[P]$ is the set of all people who have the same mother as P.

4. $\equiv$ <u>is</u> an equivalence relation. Since $m-m=0$ is even, for all $m \in \mathbb{Z}$, $\equiv$ is <u>reflexive</u>. If $m-n$ is even, then it's negative $n-m$ is also even, so $\equiv$ is <u>symmetric</u>. Suppose $m-n$ and $n-p$ are even, i.e. $m-n = 2\cdot k_1$ and $n-p = 2\cdot k_2$, for some $k_1, k_2 \in \mathbb{Z}$, so $m-p = 2(k_1+k_2)$ is even, so $m \equiv p$. So $\equiv$ is <u>transitive</u>.

8. A) $\sim$ is <u>symmetric</u>, but <u>not reflexive</u> ($m-m=0$ is not odd, so $m \not\sim m$)), <u>nor</u> <u>transitive</u>. Suppose $m-n$ and $n-p$ are both odd, then $m-p = 2\cdot k$ for some $k \in \mathbb{Z}$, so even. Why? Suppose $m-n = 2k_1+1$ and $n-p = 2k_2+1$ (so odd numbers) then by substitution we have $m-p = 2k_1+2k_2+1+1$ or $m-p = 2(k_1+k_2+1)$.

10. A) $\sim$ is <u>reflexive</u> for any $(m,n) \in \mathbb{N}\times\mathbb{N}$, $m+n=n+m$. so $(m,n)\sim(m,n)$.

$\sim$ is <u>symmetric</u>, Suppose $(m,n)\sim(k,l)$, then $m+l = n+k$. But then $k+n = l+m$, so $(k,l)\sim(m,n)$

$\sim$ is <u>transitive</u>. Suppose $(m,n)\sim(k,l)$ and $(k,l)\sim(p,q)$ then $m+l = n+k$ and $k+q = l+p$. $(l = k+q-p)$, so $m+(k+q-p) = n+k$, therefore $m+q = n+p$, so $(m,n)\sim(p,q)$.

Supplemental Problems

1\. A) $X_n = 3X_{n-1} + 4X_{n-2}$, $n \geq 2$, where $X_0 = 0$, $X_1 = 1$.

Characteristic eqn: $\alpha^2 - 3\alpha - 4 = 0$, so roots are $4, -1$.

General Solution: $X_n = C_1 4^n + C_2(-1)^n$, for all $n \geq 0$.

Solve for C_1, C_2:

$$\begin{aligned} X_0 &= 0 = C_1 + C_2 \\ X_1 &= 1 = 4C_1 - C_2 \end{aligned} \Longrightarrow C_1 = 1/5,\ C_2 = -1/5$$

Exact solution: $X_n = 1/5\, 4^n - 1/5(-1)^n$, $n \geq 0$.

B). $X_n = 7X_{n-1} - 16X_{n-2} + 12X_{n-3}$, $n \geq 3$, $X_0 = 1$, $X_1 = 4$, $X_2 = 10$.

Characteristic eqn: $\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$.

roots . $2, 2, 3$ $\leftarrow$ $(\alpha-2)(\alpha-2)(\alpha-3) = 0$.

Repeated roots! So

General solution: $X_n = C_1 3^n + C_2 n 2^n + C_3 2^n$.

Solve for C_1, C_2, C_3:

$$\begin{aligned} X_0 &= 1 = C_1 + C_3 \\ X_1 &= 4 = 3C_1 + 2C_2 + 2C_3 \\ X_2 &= 10 = 9C_1 + 8C_2 + 4C_3 \end{aligned} \Longrightarrow \begin{aligned} C_1 &= -2 \\ C_2 &= 2 \\ C_3 &= 3. \end{aligned}$$

So Exact Solution: $X_n = -2 \cdot 3^n + n \cdot 2^{n+1} + 3 \cdot 2^n$.

c) $X_n = 2X_{n-1} + n$, $n \geq 1$, $X_0 = 1$.

This is first-order non-homogeneous recurrence relation. In order to use Characteristic equations, we must turn this into a homogeneous equation.

First, we have

(1) $x_n - 2x_{n-1} = n$.

Look at the terms x_{n+2} and x_{n+1}.

(2) $x_{n+2} - 2x_{n+1} = n+2$

(3) $x_{n+1} - 2x_n = n+1$

multiply (3) by 2 and subtract from 2, yields

(4) $x_{n+2} - 4x_{n+1} + 4x_n = -n$.

Now add (1) to (4). This yields

(5) $x_{n+2} - 4x_{n+1} + 5x_n - 2x_{n-1} = 0$.

$\Rightarrow$ Homogeneous! Characteristic eqn:

$\alpha^3 - 4\alpha^2 + 5\alpha - 2 = 0$. This factors to

$(\alpha-1)(\alpha-1)(\alpha-2) = 0$, so roots are 1, 1, 2.

the general solution for this recurrence is

$x_n = c_1 2^n + c_2 n \cdot 1^n + c_3 \cdot 1^n$.

or $x_n = c_1 2^n + c_2 n + c_3$

Solve for c_1, c_2, c_3:

$$\begin{aligned} x_0 &= 1 = c_1 + c_3. \\ x_1 &= 3 = 2c_1 + c_2 + c_3. \\ x_2 &= 8 = 4c_1 + 2c_2 + c_3 \end{aligned} \Rightarrow \begin{aligned} c_1 &= 3 \\ c_2 &= -1 \\ c_3 &= -2 \end{aligned}$$

So <u>exact solution</u>:

$x_n = 3 \cdot 2^n - n - 2$, for all $n \geq 0$

D) $X_n = 2X_{n/2} + n \cdot \log_2 n$, n is a power of 2, $X_1 = 0$.

non-linear recurrence eqn. so transform the domain ($k = \log_2 n$) so this yields

$$X_k = 2X_{k-1} + k \cdot 2^k \quad \text{or} \quad X_k - 2X_{k-1} = k \cdot 2^k \quad (1)$$

Now its linear, but inhomogeneous.

looking at X_{k+2} and X_{k+1}, yields:

$$X_{k+2} - 2X_{k+1} = (k+2)2^{k+2} \quad (2)$$
$$X_{k+1} - 2X_k = (k+1)2^{k+1} \quad (3)$$

Multiply (3) by 4 and subtract from (2).:

$$X_{k+2} - 6X_{k+1} + 8X_k = -k2^{k+2} \quad (4)$$

multiply (1) by 4, add to (4). gives us

$$X_{k+2} - 6X_{k+1} + 12X_k - 8X_{k-1} = 0.$$

Homogeneous, with characteristic eqn.:

$$\alpha^3 - 6\alpha^2 + 12\alpha - 8 = 0. \Rightarrow (\alpha - 2)^3 = 0.$$

So <u>General Solution</u>:

$$X_k = C_1 2^k + C_1 k 2^k + C_3 k^2 2^k.$$

<u>Transform</u> back to n, yields

$$X_n = C_1 n + C_2 n \log_2 n + C_3 n \log_2^2 n.$$

Now use initial condition and (1) to solve for C_1, C_2, C_3.