Fall 2003 Midterm Solutions

1. A : Row-reduce the augmented matrix of the system, the final row will be $[0\ 0\ 0\ \ -b_1+b_2+b_3]$. By the existence and uniqueness theorem, the system is consistent if and only if the final entry in that row equals zero.

2. D : The leading entries don't move from left to right as we go down the matrix.

3. B : You can either use trial and error (i.e. just see which of the given vectors gets transformed to $\mathbf{0}$ when you multiply by A - this is probably quicker) or set up an augmented matrix (add a column of zeroes to A), row reduce, find that the general solution is $x_1 = (3/2)x_2$, x_2 free, then see which of the vectors has this form.

4. D : Questions about one-to-one and onto are answered by looking at the pivot positions. So row-reduce the given matrix, and discover that it has no pivot in the third column (so it is not one-to-one), but does have a pivot in every row (so it is onto).

5. B : We want to know what is the dimension of the solution set. The answer is, the dimension of the solution set is equal to the number of free variables. So, row reduce the augmented matrix $[A\ \mathbf{0}]$ and count how many free variables you have. You get just one, so the solution set is one-dimensional, i.e. a line.

6. C : This is a lot like 1. Row reduce the given matrix, and find that the last row becomes $[0\ 0\ 0\ h-4]$. The existence and uniqueness theorem says that the system will be consistent if and only if the last entry in this row equals zero, so we want $h = 4$.

7. D : This one is a bit tricky, in that statements (b) and (c) are somewhat complicated. However, statement (d) is obviously not always true, because we know (for example, from question 4) that not every linear transformation is one-to-one.

8. C : There are two ways to do this; the first is to compute the inverse by row-reducing the matrix $[A\ I_3]$, and reading off the second row. The second way is to reason as follows: since $A^{-1}A = I_3$, we know in particular that the entries in the second row of I_3 (namely $[0\ 1\ 0]$) are the dot-products of the second row of A^{-1} by the columns of A. So just check each of these vectors in turn, and see which one has dot product 0 with the first column of A, dot product 1 with the second column of A, and dot product 0 with the third column of A.

9. B : This is similar to an "additional" question you did in homework. To find the standard matrix for T, you want to know $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. You're given two equations involving these quantities; so attack the problem as if you were solving a system of two equations in two variables. More explicitly, first use the fact that T is linear to rewrite the left-hand side of the first equation as $T(\mathbf{e}_1) + T(\mathbf{e}_2)$, and the to rewrite the left-hand side of the second equation as $T(\mathbf{e}_1) - T(\mathbf{e}_2)$. To find $T(\mathbf{e}_1)$, add the two equations together and divide by 2. To find $T(\mathbf{e}_2)$, subtract the second equation from the first and divide by 2.

10. A : Again, this is similar to a problem from the additional homework. Use the fact that T is linear to rewrite $T(2\mathbf{u} - 3\mathbf{v})$ as $2T(\mathbf{u}) - 3T(\mathbf{v})$, and then substitute in the values of $T(\mathbf{u})$ and $T(\mathbf{v})$ that you're given.

11. C : You can immediately rule out (a), because of the presence of the zero vector. Similarly, one of the vectors in (d) is a multiple of another, so these can't be linearly independent either. You're left with just (b) and (c). Now for each of these options, form an augmented matrix whose columns are the vectors in question, and then the zero vector (you're looking for nontrivial solutions of a certain homogeneous linear equation - see your notes or the book). You'll find that the matrix

you get from (c) reduces very quickly to row-echelon form, and from this form it is clear that there are no free variables, hence no nontrivial solutions. So those vectors are LI.

12. C : To find the standard matrix of a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, you need to compute the images of $\mathbf{e}_1$ and $\mathbf{e}_2$. Draw a picture. Rotate $\mathbf{e}_1$ counterclockwise by $\pi/4$ - it moves to $(1/\sqrt{2}, 1/\sqrt{2})$. Then reflect this vector through the line $x_2 = x_1$ - but this vector lies on this line, so the reflection doesn't affect it. So $T(\mathbf{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Do the same process to $\mathbf{e}_2$ (this time the reflection does have an effect).

13. D : The vector $\mathbf{u}$ is in the span of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the linear system with augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{u}]$ is consistent (because a solution to this system gives the weights needed to write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$). So set up this matrix and row-reduce. You'll get the last row looking like $[0\ 0\ h-16]$, and then the system is consistent iff $h = 16$.

14. C : Remember to do the operation inside the parentheses first. So compute AB, and then take its transpose.

15. D : (a) and (c) are properties we learned in class. (b) is somewhat tricky, but is also true (use the distributive laws to expand $(A+B)(A+B)$, and then use the fact that A and B commute). So it must be (d). Indeed, we saw an example of this in one of the additional homework problems.

16. B : "The inverse of $7A$ is B" means that $B(7A) = I_3$. The constant 7 can be pulled out the front, and the parentheses rearranged, to get $(7B)A = I_3$. This says that the inverse of A is $7B$.

17. A : Again, this is like one of your additional homework problems (are you noticing a trend?). (b) is not linear, because of the added constants. (c) is not linear, because of the trigonometric functions. (d) is not linear, because of the square. So it must be (a).

18. D : Questions about one-to-one'ness are answered by looking at pivot columns of the standard matrix. To find the standard matrix of this transformation, plug in $x_1 = 1,\ x_2 = 0$ to find $T(\mathbf{e}_1)$, and then plug in $x_1 = 0,\ x_2 = 1$ to find $T(\mathbf{e}_2)$. Then arrange these into a matrix: $\begin{bmatrix} 2 & -8 \\ -4 & h \\ 1 & 12-h \end{bmatrix}$. Row-reduce this, and see which value of h will ensure that you have a pivot in each column.

19. B : We saw in a theorem in class (from section 1.4) that an $m \times n$ matrix has a pivot in every row if and only if the columns of that matrix span $\mathbb{R}^m$, so the answer must be (b). (a) is false, because the columns of an $m \times n$ matrix each have m entries, so the columns don't lie in $\mathbb{R}^n$. (c) might be false, if the matrix has more columns than rows. (d) might also be false, if the matrix has as many columns as it has rows.

20. D : A 2×2 matrix is not invertible if and only if its determinant equals zero. The determinant of this matrix is $4k + 6$. Solve for k.