Math 220 Sections 1, 9 and 11. Review Sheet 1

October 15, 2006

1.1 Systems of Linear Equations

Theory:

Key terms and ideas - you need to know what they mean, and how they fit together:

- Linear equation in the variables $x_1, x_2, \ldots, x_n$; System of linear equations
- Solution of a linear system; Solution set of a linear system (NB these are *not* the same thing)
- Equivalent systems
- Consistent/Inconsistent systems
- Coefficient matrix of a linear system; Augmented matrix of a linear system
- Row operations; Row-equivalent matrices

Key facts - you need to know them (and have an idea of why they are true):

- Every linear system has either (i) No solutions; (ii) One unique solution; or (iii) infinitely many solutions.
- Two linear systems are equivalent if and only if their augmented matrices are row-equivalent.

Practice:

Things you need to be able to do/answer:

- "Which of the following equations are linear?" - to answer this, look for any expression involving variables being multiplied, squared, $\log$'ed, $\sin$'ed, etc.
- "What is the augmented matrix of this system?" - this kind of question is a gift, as long as you remember to arrange all the variables in the same order, and move the constants over to the right-hand side.

1.2 Row Reduction and Echelon Forms

Theory:

Key terms and ideas:

- Row Echelon Form (aka REF, Echelon Form); Reduced Row Echelon Form (aka RREF, Reduced Echelon Form)
- Leading entry; Nonzero row/column; Pivot position; Pivot column
- Basic variable; Free variable
- General solution

Key facts:

- Each matrix is row-equivalent to one and only one matrix in RREF.
- Existence and uniqueness theorem: *A linear system is consistent if and only if the RREF of its augmented matrix has no row of the form* $[0\ 0\ \ldots\ 0\ b]$ *where* $b \neq 0$*. If a system is consistent, then it has a unique solution if and only if there are no free variables.* Virtually everything else we've seen in the course is based on this theorem.

Practice:

Things you need to be able to do/answer:

- "Row reduce this matrix." You absolutely *need* to be able to do this quickly, accurately and confidently. Following the algorithm given in class is a good way to make sure you don't go round in circles.
- "Which of these matrices is in REF/RREF?" You need to remember the definitions or row echelon and reduced row echelon forms.
- "Write the general solution of this particular system." The procedure is:
  1. Write the augmented matrix
  2. Row reduce it (down to REF)
  3. Decide if the system is consistent
  4. If it is, keep row reducing (down to RREF)
  5. Decide which variables (if any) are free
  6. Rewrite the matrix as a set of equations, writing each basic variable in terms of the free variables.
- "For which values of h is this system consistent", and "For which h does the system have a unique solution?" To answer these, follow the above procedure and see which values of h (if any) will allow you to avoid rows like $[0 \ldots 0\, b]$ (or alternatively, free variables). This usually requires solving some equation involving h.

1.3 Vector Equations

Theory:

Key terms and ideas:

- Vector, $\mathbb{R}^2$, $\mathbb{R}^n$
- Definitions of the sum of two vectors, and the product of a scalar and a vector
- Geometric descriptions of $\mathbb{R}^2$, $\mathbb{R}^3$
- Parallelogram law for addition, dilating/contracting a vector by scalar multiplication
- Linear combinations, weights
- Vector equation
- $\mathrm{span}\{\mathbf{v}_1 \ldots, \mathbf{v}_p\}$

Key facts:

- "Algebraic properties of $\mathbb{R}^n$" - see Page 32.
- A vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $[\mathbf{a}_1 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$.
- As a consequence of the above, a vector $\mathbf{b}$ belongs to the span of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system with augmented matrix $[\mathbf{a}_1 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$ is consistent.

Do you really understand span? If not, you should really make it a priority to get on top of this idea. One way to think of it is like this: you're given some vectors $\mathbf{v}_1, \ldots \mathbf{v}_n$, and you have a way to make new vectors out of these old ones (i.e. form linear combinations). Then $\mathrm{span}\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is the collection of all those vectors you can make from $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by taking linear combinations.

Practice:

Things you need to be able to do/answer:

- Basic algebraic operations on vectors: eg "add these vectors", etc.
- "Does $\mathbf{b}$ belong to the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$?" To answer this, row reduce the matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n \ \mathbf{b}]$ and see if the system is consistent.
- "For which value(s) of h does this vector $\mathbf{b}$ (which might depend on h) belong to the span of these other vectors (some of which might also depend on h)?" To answer this, set up the matrix as above, row reduce, and see which values of h (if any) will make the system consistent.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Theory:

Key terms and ideas:

- Definition of the matrix-vector product
- Matrix equation
- Row-vector rule for computing $A\mathbf{x}$

Key facts:

- Let A be a matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation $x_1\mathbf{a}_1 + \cdots x_n\mathbf{a}_n = \mathbf{b}$ and as the linear system whose augmented matrix is $[\mathbf{a}_1 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$. This says that linear systems, vector equations and matrix equations are just three ways of writing the same thing.
- For some particular vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.
- Let A be an $m \times n$ matrix. The following statements are equivalent:
  1. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$;
  2. The columns of A span $\mathbb{R}^m$;
  3. A has a pivot position in every row.
- "Properties of the matrix-vector product" - see page 45.

Practice:

Things you need to be able to do/answer:

- "Translate this particular vector equation/matrix equation/linear system into a matrix equation/linear system/vector equation." To do this, you just need to remember the way in which these three things are related. When you write an augmented matrix, don't forget that the final column should come from some constant vector, usually called $\mathbf{b}$, or sometimes $\mathbf{0}$.
- "If A is the given matrix and $\mathbf{b}$ is an arbitrary vector (with entries $b_1, \ldots, b_m$), then for which $\mathbf{b}$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution?" To solve this, row-reduce the augmented matrix $[A \ \mathbf{b}]$, and see which values of $b_1, \ldots, b_m$ make the system consistent (this will usually involve writing down an equation involving the b's).
- "For this particular matrix A, which depends somehow on h, for what values of h do the columns of A span $\mathbb{R}^m$?" This is just one of many possible questions of this kind. To answer it, use the theorem that says "the following statements are equivalent". In this example you want to know about statement (2), and the theorem says that you can answer this by seeing whether or not (3) is true. i.e., row reduce the matrix and see which values of h give you a pivot position in every row.

1.5 Solution Sets of Linear Systems

Theory:

Key terms and ideas:

- Homogeneous vs nonhomogeneous systems
- Nontrivial solution of a homogeneous system
- Parametric vector form
- Translations of solution sets; parallel lines/planes

Key facts:

- Homogeneous systems *always* have a solution: the zero vector.
- When you write the solution set of a system in parametric form, the number of parameters (i.e. free variables) equals the dimension of the solution set.
- Theorem: *Suppose the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent for some given* $\mathbf{b}$. *Then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{p} + \mathbf{v}_h$, *where* p *is a particular solution to* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{v}_h$ *is any solution to the homogeneous equation* $A\mathbf{x} = \mathbf{0}$. You should also remember the pictures that go with this theorem (pages 53 and 54).

Practice:

Things you need to be able to do/answer:

- "Write the solution set to this given equation (might be a vector equation, matrix equation or linear system) in parametric vector form." To do this,

  1. First write the corresponding augmented matrix;
  2. Row-reduce, and write the general solution (as you did in section 1.2);
  3. Write the general solution as a vector: this vector will have as many entries as there are variables in the system, and each entry will be given by the corresponding part of the general solution;
  4. Split this vector into a sum of several vectors: one for each free variable, and one for the constant terms (if required);
  5. Factor out the free variables, and maybe rename them.

- "Given some particular matrix A and some vector $\mathbf{b}$ (might be $\mathbf{0}$), describe geometrically the solution set of the equation $A\mathbf{x} = \mathbf{b}$." To do this, first get the parametric vector form of the solution, as above. The number of parameters (free variables) gives you the dimension: none $\rightarrow$ point, one $\rightarrow$ line, two $\rightarrow$ plane, three $\rightarrow$ $\mathbb{R}^3$. If there is no constant vector in the PVF, then the solution set contains (aka passes through) the origin. If the PVF has a constant vector $\mathbf{p}$, then the solution set passes through $\mathbf{p}$.

1.7 Linear Independence

Theory:

Key terms and ideas:

- Linear Independence (LI); Linear Dependence (LD); Linear Dependence Relation (LDR)

Key facts:

- The columns of a matrix A are LI if and only if the equation $A\mathbf{x} = \mathbf{0}$ has a no non-trivial solution. This is the case if and only if every column of A is a pivot column.
- A set of one vector is LI if and only if it isn't the zero vector.
- A set of two vectors is LI if and only if neither is a multiple of the other.
- Any set containing the zero vector is LD.
- Any set of vectors containing more vectors than the number of entries in each vector, is LD. (eg a set of 3 vectors in $\mathbb{R}^2$ must be LD.)
- A set of vectors is LD if and only if one of the vectors can be written as a linear combination of the others.
- The span of k linearly independent vectors is a k-dimensional space. eg the span of two LI vectors is a plane.

Practice:

Things you need to be able to do/answer:

- "Are these vectors LI/LD? If they're LD, find a LDR." First, remember our general time-saving facts: eg, if there are more vectors than entries in each vector, the vectors are LD. If there are two vectors, then just check to see if one is a multiple of the other. If none of the tricks are applicable, remember that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are LI iff the matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$ has a pivot in every column. So row reduce this matrix.
- "Given certain vectors, give a geometric description of their span." The dimension of $\mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is equal to the number of pivot positions in the matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$. So row-reduce this matrix.

1.8 Linear Transformations

Theory:

Key terms and ideas:

- Transformation; domain; codomain
- Image; range
- Matrix transformation (i.e. a transformation defined by multiplying by a matrix)
- Linear transformation

Key facts:

- Every matrix transformation is linear.
- A transformation T is linear if and only if $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars c and d. This fact is *absolutely crucial!!*
- The image of $\mathbf{0}$ under any linear transformation is $\mathbf{0}$.

Practice:

Things you need to be able to do/answer:

- "Which of the following is a linear transformation? (followed by a list of different formulae)" Strictly speaking, the way to do this is to check that $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for any choices of vectors and scalars. But, as one of my teachers back in Australia was fond of saying, "real mathematicians *know* when something is linear." As a general rule, if you see squares, $\sin$'s, products of variables, etc, the transformation is not linear. Likewise, if you see a constant being added, the transformation is not linear (because the image of $\mathbf{0}$ will not be $\mathbf{0}$).
- "What is the image of such-and-such a vector under such-and-such a transformation?" Usually this is just a case of plugging some numbers into a formula.
- "Does this particular vector $\mathbf{b}$ lie in the range of this particular linear transformation?" To solve this, find the standard matrix A of the transformation, if it's not already given to you (see the next section). Then a vector $\mathbf{b}$ lies in the range of the transformation if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.

1.9 The Matrix of a Linear Transformation

Theory:

Key terms and ideas:

- The $n \times n$ identity matrix I_n; its columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$.
- The standard matrix of a linear transformation.
- Geometric terminology, such as dilation, contraction, reflection, rotation, projection. See Example 4 on page 77, and the various tables on pages 85–87.
- One-to-one; Onto.

Key facts:

- Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a unique standard matrix, $A = [T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$ where $\mathbf{e}_i$ is the i^{th} column of I_n.
- A linear transformation is onto iff the columns of its standard matrix A span $\mathbb{R}^m$, which happens iff every row of A contains a pivot position.
- A linear transformation is one-to-one iff the columns of its standard matrix A are linearly independent, which happens iff every column of A is a pivot column.
- A linear transformation T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has a unique solution (this is a consequence of the above fact, but it is worth noting).

Practice:

Things you need to be able to do/answer:

- "Given this particular formula for a linear transformation T, find its standard matrix." To do this, just plug in the vectors $\mathbf{e}_1, \mathbf{e}_2$ etc. into the formula, then arrange their images into a matrix.
- "Given this particular geometric description of a linear transformation T (usually $T : \mathbb{R}^2 \to \mathbb{R}^2$), find its standard matrix." To do this, draw a picture, and just look at what the transformation does to $\mathbf{e}_1$ and $\mathbf{e}_2$. You might need to remember some trigonometry.
- You might be asked the reverse question: "If T has this standard matrix, describe T geometrically." To do this, look at the images of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ when you multiply them by the given matrix, and try to work out what's happening geometrically (again, draw a picture). For exam purposes, you'll be given a list of possibilities, which should make your job easier.

- "If $T(c_1\mathbf{e}_1 + c_2\mathbf{e}_2) =$ (some given vector), and $T(d_1\mathbf{e}_1 + d_2\mathbf{e}_2) =$ (some other given vector), (where c_1, c_2, d_1, d_2 are some given numbers), find the standard matrix for T." (note that the question might involve more $\mathbf{e}_i$'s, or more equations, but the idea is the same). You need to find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. To do this, use the *crucial fact* coming from the linearity of T (see above) to rearrange the equations you're given into equations involving $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. Solve these equations.
- "If T is this particular linear transformation, determine if T is one-to-one (or onto)." To do this, find the standard matrix of T as above, then row reduce and look at the pivot positions.

2.1 Matrix Operations

Theory:

Key terms and ideas:

- a_{ij} notation; diagonal entries; diagonal matrix; zero matrix.
- Sum and scalar multiple of matrices
- Multiplication of matrices: when is it defined, how is it defined.
- Powers of a matrix; transpose of a matrix

Key facts:

- Properties of sum and scalar multiples of matrices (cf. Theorem 1 on page 108)
- Properties of matrix multiplication (cf. Theorem 2 on page 113)
- Properties that *don't* hold for matrix multiplication (cf. "Warnings" on page 114)
- Properties of the transpose (cf. Theorem 3 on page 115).

Practice:

This section is all about computations: computing sums, scalar multiples, products, powers and transposes of matrices. Also, you shouldn't be scared if you see more than one of these operations combined in the same question. Just remember the properties listed in the various theorems, and take care of the order of operations (in particular, operations inside parentheses get done before those outside the parentheses).

2.2 The Inverse of a Matrix

Theory:

Key terms and ideas:

- Invertible matrix; singular matrix; the inverse of a matrix.
- The determinant of a 2×2 matrix.

Key facts:

- The idea of invertibility only applies to square matrices. Not every matrix is invertible.
- A 2×2 matrix is invertible if and only if its determinant is nonzero. If it is invertible, there's a simple formula for the inverse (and you should remember this formula) .
- If A is invertible, then the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for each vector $\mathbf{b}$.
- If A and B are invertible: $(A^{-1})^{-1} = A$; $(AB)^{-1} = B^{-1}A^{-1}$; $(A^T)^{-1} = (A^{-1})^T$.
- A is invertible if and only if it is row-equivalent to I_n, and any sequence of row operations which row-reduces A will transform I_n to A^{-1}.

Practice:

Things you need to be able to do/answer:

- "Determine if this 2×2 matrix is invertible, and if it is compute its inverse." To do this, first calculate the determinant $ad - bc$. If it's nonzero, the matrix is invertible, and the inverse is obtained by swapping the diagonal entries, negating the off-diagonals, and dividing everything by the determinant.
- "Suppose A and B are invertible. Simplify the following expression (something like $(A^{-1})^T(BA)^T(B^T)^{-1}$)" [1] To do this, use the properties of the inverse (Theorem 6 on page 121) along with properties of the transpose, etc (see the previous section). Remember that the product of anything by its inverse is I, and that just like the number 1, the matrix I can be removed from any multiplicative expression.
- "Calculate the inverse of this $n \times n$ matrix A (where $n > 2$; typically $n = 3$.)" To do this, row-reduce the matrix $[A \mid I_n]$. When the left-hand side looks like I, the right-hand side will be A^{-1}. Check your answer by multiplying A^{-1} by A and making sure you get I_n.

[1] Incidentally, this is equal to I.

2.3 Characterisations of Invertible Matrices

Theory:

Key terms and ideas:

- Invertible linear transformation.

Key facts:

- The Invertible Matrix Theorem (Theorem 8 on page 129). This brings together most of the ideas we've seen so far. The idea is not to memorise this theorem (although I suppose this wouldn't hurt, if you're the kind of person who likes to memorise things), but more importantly you should be able to go through it and understand why all of the statements are equivalent. The key idea is that if A has an inverse, then A can be "cancelled" out of equations by multiplying by the inverse.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if its standard matrix A is invertible. In this case, the standard matrix of T^{-1} is A^{-1}.

Practice:

Things you need to be able to do/answer: This section is all about theory. A typical question would be, "Which of the following statements is true/false?" followed by a list of statements about invertible matrices, linear transformations, pivot positions, etc. To answer such a question, read through the statements one by one, and see which one(s) must be true, based on the invertible matrix theorem (and other theorems we've seen). There's really not much more to it than that.