# Midterm I. Math 230. Spring2005

4, October, 2006

**Problem 1** *Pick a point on the first line. Say,* $t = 0$ *yields* $A(0,1,0) \in L_1$. *Pick a point on the second line. Say,* $s = 0$ *yields* $B(1,4,0) \in L_2$. *Take any direction vector of either line, say,* $\overrightarrow{d} = \langle 2,-3,1\rangle$ *is the direction vector of the second line. Vectors* $\overrightarrow{d}$ *and* $\overrightarrow{AB}$ *are parallel to the plane of the two lines, but not to each other. Therefore,* $\overrightarrow{n} = \overrightarrow{d} \times \overrightarrow{AB}$ *is orthogonal to the plane, and so can be taken as its normal vector.* $\overrightarrow{n} = \overrightarrow{d} \times \overrightarrow{AB} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -3 & 1 \\ 1 & 3 & 0 \end{vmatrix} = \langle -3,1,9\rangle$.
*To define a plane we also need some point on it, say,* $A(0,1,0)$, *so the equation is* $-3(x-0)+1(y-1)+9(z-0)=0$, *or just* $-3x+y+9z-1=0$.

**Problem 2** $z = \rho\cos(\phi) = 2\cos(\frac{\pi}{6}) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$
$r = \rho\sin(\phi) = 2\sin(\frac{\pi}{6}) = 2\frac{1}{2} = 1$
$x = r\cos(\theta) = 1\cos(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$
$y = r\sin(\theta) = 1\sin(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$
*So the answer to (a) is* $(\sqrt{3}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
$\rho^2\sin^2(\phi) = (\rho\sin(\phi))^2 = r^2 = x^2+y^2$
$\rho^2\cos^2(\phi) = (\rho\cos(\phi))^2 = z^2$
*So the answer to (b) is* $2(x^2+y^2) - z^2 = 1$.

**Problem 3** $\overrightarrow{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2}\rangle$
$|\overrightarrow{r}'(t)| = \sqrt{e^{2t}+e^{-2t}+2} = \sqrt{(e^t+e^{-t})^2} = e^t+e^{-t}$
*Hence, the unit tangent vector* $\overrightarrow{T}(t) = \frac{1}{e^t+e^{-t}}\langle e^t, -e^{-t}, \sqrt{2}\rangle$
*Its derivative can be found applying the product rule (derivative of a product of a scalar function and a vector function):*
$\overrightarrow{T}'(t) = \frac{-(e^t-e^{-t})}{(e^t+e^{-t})^2}\langle e^t,-e^{-t},\sqrt{2}\rangle + \frac{1}{e^t+e^{-t}}\langle e^t, e^{-t}, 0\rangle = \frac{-(e^t-e^{-t})}{(e^t+e^{-t})^2}\langle e^t,-e^{-t},\sqrt{2}\rangle + \frac{1}{(e^t+e^{-t})^2}\langle e^{2t}+1, 1+e^{-2t}, 0\rangle = \langle -e^{2t}+1, 1-e^{-2t}, -\sqrt{2}e^t+\sqrt{2}e^{-t}\rangle + \frac{1}{(e^t+e^{-t})^2}\langle e^{2t}+1, 1+e^{-2t}, 0\rangle = \frac{1}{(e^t+e^{-t})^2}\langle 2,2,-\sqrt{2}e^t+\sqrt{2}e^{-t}\rangle$
*We need to normalize* $\overrightarrow{T}'(t)$ *in order to find the unit normal vector.* $|\overrightarrow{T}'(t)| = \frac{1}{(e^t+e^{-t})^2}\sqrt{4+4+2(e^{-t}-e^t)^2}$, *so the unit normal vector*
$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|} = \frac{1}{\sqrt{8+2(e^{-t}-e^t)^2}}\langle 2,2,-\sqrt{2}e^t+\sqrt{2}e^{-t}\rangle$

*To find the curvature,* $\kappa(t) = \frac{|\overrightarrow{T}'(t)|}{|\overrightarrow{r}'(t)|} = \frac{\frac{1}{(e^t+e^{-t})^2}\sqrt{4+4+2(e^{-t}-e^t)^2}}{e^t+e^{-t}} = \frac{1}{(e^t+e^{-t})^3}\sqrt{8+2(e^{-t}-e^t)^2}$

**Problem 4** $\overrightarrow{r}'(t) = \langle t, \sqrt{2t}, 1\rangle$

$|\overrightarrow{r}'(t)| = \sqrt{t^2+2t+1} = t+1$, *as far as* $1 \le t \le 3$, *so* $t+1 \ge 0$. *Finally,* $L = \int_1^3 |\overrightarrow{r}'(t)|dt = \int_1^3 (t+1)dt = (t^2/2+t)\Big|_1^3 = (3^2/2+3) - (1^2/2+1) = 6$

**Problem 5** $\overrightarrow{v}(t) = \overrightarrow{v}(0) + \int_0^t \overrightarrow{a}(u)du = \langle 0,1,2\rangle + (\langle -\sin(u), \cos(u), u^2 - u\rangle\Big|_0^t) = \langle 0,1,2\rangle + (\langle -\sin(t), \cos(t), t^2-t\rangle - \langle 0,1,0\rangle) = \langle -\sin(t), \cos(t), t^2-t+2\rangle$

$\overrightarrow{r}(t) = \overrightarrow{r}(0) + \int_0^t \overrightarrow{v}(u)du = \langle 1,0,0\rangle + (\langle \cos(u), \sin(u), \frac{u^3}{3} - \frac{u^2}{2} + 2u\rangle\Big|_0^t) = \langle 1,0,0\rangle + (\langle \cos(t), \sin(t), \frac{t^3}{3} - \frac{t^2}{2} + 2t\rangle - \langle 1,0,0\rangle) = \langle \cos(t), \sin(t), \frac{t^3}{3} - \frac{t^2}{2} + 2t\rangle$

$|\overrightarrow{v}(1)| = |\langle -\sin(1), \cos(1), 1^2 - 1 + 2\rangle| = \sqrt{(-\sin(1))^2 + (\cos(1))^2 + (2)^2} = \sqrt{5}$

**Problem 6** *We did not cover this material*

**Problem 7** *We did not cover this material*

**Problem 8** *Extensive computations. Idea: substitute one equation into the other.* $x + y + x^2 + 3y^2 = 4$, *or* $(x+1/2)^2 + 3(y+1/6)^2 = 13/3$. *This projection onto the xy-coordinate plane is an ellipse, which can be parametrized by* $t$. *Then find the tangent vector (i.e. the derivative of the vector function, describing this parametrization). Finally, substitute the value of* $t$ *yielding* $(1, -1, 4)$. *This value of* $t$ *can be found from the parametrization of the ellipse mentioned above, i.e. from the first coordinate.* $t = \cos^{-1}(...)$.

**Problem 9** *True: a, e (check the dot product), h (think of the volume of the corresponding parallelepiped), i*

*False: b (because they may lie on one line), c, d(in our definition curvature is always nonnegative), f (the center is* $(2, -3, 5)$*), j (*$f_{xy}$ *means the opposite, but you do not need to know that for the exam)*

*Statement **g** makes no sense, because this is a cross product of a vector with a scalar.*

**Problem 10** *The question is in effect to find the tangential component of the acceleration.* $\overrightarrow{v}(t) = \overrightarrow{r}'(t) = \langle \frac{1}{t}, 2t, 2t\rangle$

$\overrightarrow{a}(t) = \overrightarrow{v}'(t) = \langle \frac{-1}{t^2}, 2, 2\rangle$

$proj_{\overrightarrow{v}(t)}\overrightarrow{a}(t) = \frac{\overrightarrow{a}(t)\cdot\overrightarrow{v}(t)}{|\overrightarrow{v}(t)|^2}\overrightarrow{v}(t) = \frac{\frac{-1}{t^3}+4t+4t}{\frac{1}{t^2}+4t^2+4t^2}\langle \frac{1}{t}, 2t, 2t\rangle = \frac{8t^4-1}{8t^4+1}\langle \frac{1}{t^2}, 2, 2\rangle$