Math 230, Fall 2006

Solutions to Midterm Exam 1

Problem 1.

1. Find an equation of the sphere with center $P = (1, 1, 2)$ and passing through $Q = (2, 1, 1)$.

2. Find the intersection points (if any) of this sphere and the z-axis.

3. Find the intersection points (if any) of this sphere and the x-axis.

Solutions.

1. $\overrightarrow{PQ} =< 1, 0, -1 >$ and $|\overrightarrow{PQ}| = \sqrt{2}$.

   Hence, an equation of the sphere with center $P = (1, 1, 2)$ and passing through $Q = (2, 1, 1)$ is $(x - 1)^2 + (y - 1)^2 + (z - 2)^2 = 2$.

2. Setting $x = 0$ and $y = 0$, we get

$$(-1)^2 + (-1)^2 + (z - 2)^2 = 2 \quad \text{or} \quad (z - 2)^2 = 0.$$

   The solution of this equation is $z = 2$. So, the intersection point of the sphere and the z-axis is $(0, 0, 2)$.

3. Similarly, setting $y = z = 0$, we get

$$(x - 1)^2 + (-1)^2 + (-2)^2 = 2 \quad \text{or} \quad (x - 1)^2 = -3.$$

   There is no solution for this equation. Consequently, the sphere does not intersect the x-axis.

Problem 2. Find the volume of the parallelepiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ if $P = (1, 1, 2)$, $Q = (2, 3, 1)$, $R = (-1, 1, 5)$ and $S = (1, 8, -2)$.

Solution. The volume of this parallelepiped is given by

$$V = |\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|.$$

Since

$$\overrightarrow{PQ} =< 1, 2, -1 > \quad \overrightarrow{PR} =< -2, 0, 3 >, \quad \text{and} \quad \overrightarrow{PS} =< 0, 7, -4 >$$

one gets

$$\overrightarrow{PR} \times \overrightarrow{PS} = -21\mathbf{i} - 8\mathbf{j} - 14\mathbf{k}.$$

Therefore, $V = |-21 - 16 + 14| = |-23| = 23$

Problem 3. If the spherical coordinates of a point P are $(\rho, \theta, \phi) = (\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$,

1. find its rectangular coordinates.
2. Find its cylindrical coordinates.

Solutions.

1. Since $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, one gets $(x, y, z) = (0, 1, 1)$.
2. But $r = \sqrt{x^2 + y^2} = 1$, hence $(r, \theta, z) = (1, \frac{\pi}{2}, 1)$.

Problem 4.

1. Find parametric equations of the line segment from $P = (1, -1, 2)$ to $Q = (3, 2, 1)$.
2. Find the intersection point (if any) of the xy-plane and the line through P and Q.

Solution.

1. We have $\overrightarrow{PQ} =< 2, 3, -1 >$ hence parametric equations of the line segment from P to Q are:

$$x = 1 + 2t, \quad y = -1 + 3t, \quad z = 2 - t, \quad \text{with} \quad 0 \le t \le 1.$$

2. Parametric equations of the line through P and Q are:

$$x = 1 + 2t, \quad , y = -1 + 3t, \quad z = 2 - t, \quad \text{where} \quad t \text{ varies in } \mathbb{R}.$$

Setting $z = 0$, one gets $t = 2$. Replacing t by this value, one obtains $(x, y, z) = (5, 5, 0)$ which is the intersection point of the xy-plane and the line through P and Q.

Problem 5. A moving particle starts at the origin with initial velocity $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. If its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 12t^2\mathbf{j} + 6t\mathbf{k}$, find its position function.

Solution. One gets

$$\mathbf{v}(t) = \mathbf{r}'(t) = \int_0^t \mathbf{a}(u)du + \mathbf{v}(0) = (2t^2 + 2)\mathbf{i} + (4t^3 - 1)\mathbf{j} + (3t^2 + 1)\mathbf{k}.$$

It follows that

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(u)du + \mathbf{r}(0) = (\frac{2}{3}t^3 + 2t)\mathbf{i} + (t^4 - t)\mathbf{j} + (t^3 + t)\mathbf{k}.$$

Problem 6. Find symmetric equations of the **tangent line** to the curve given by the parametric equations

$$x = t^2 + 3t \quad y = t, \quad z = t^4 + \sin t$$

at the origin $O = (0, 0, 0)$.

Solution. One has

$$\mathbf{r}'(t) =< 2t + 3, 1, 4t^3 + \cos t > . \quad \text{Therefore,} \quad \mathbf{r}'(0) =< 3, 1, 1 > .$$

Symmetric equations of the **tangent line** to the curve at the origin are:

$$\frac{x}{3} = y = z.$$

Problem 7. Given the vector function $\mathbf{r}(t) =< \sin t - t\cos t,\ \cos t + t\sin t,\ -1 >$, find the length of its curve for $0 \leq t \leq 2$.

Solution. We have $\mathbf{r}'(t) =< t\sin t,\ t\cos t, 0 >$ and $|\mathbf{r}'(t)| = \sqrt{t^2} = t$ since $0 \le t \le 2$. Therefore,

$$L = \int_0^2 |\mathbf{r}'(t)|dt = \left[\frac{t^2}{2}\right]_0^2 = 2.$$

Problem 8. Find an equation of the plane through $P = (1, 3, 2)$, $Q = (2, 4, 3)$, and $R = (0, 3, 0)$.

Solution. We have

$$\overrightarrow{PQ} =< 1, 1, 1 > \quad \overrightarrow{PR} =< -1, 0, -2 > \quad \overrightarrow{PQ} \times \overrightarrow{PR} =< -2, 1, 1 > .$$

An equation of the plane through P, Q and R is

$$-2(x-1) + (y-3) + (z-2) = 0 \quad \text{or} \quad -2x + y + z = 3.$$

Problem 9. Suppose

$$\mathbf{r}(t) =< t,\ \frac{t^2}{2},\ \frac{t^3}{3} >$$

is the position function for a moving particle P at time t. Find the normal component a_N of the acceleration at $t = 1$.

Solution. One gets

$$\mathbf{r}'(t) =< 1,\ t,\ t^2 >, \quad \mathbf{r}''(t) =< 0,\ 1,\ 2t > \quad \text{and} \quad \mathbf{r}'(1)\times\mathbf{r}''(1) =< 1, -2, 1 > .$$

Hence,

$$a_N = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

Problem 10. Find the curvature of the curve C given by

$$\mathbf{r}(t) =< 4t,\ \sin t + 2\cos t,\ 2\sin t - \cos t >$$

at $P = (0, 2, -1)$.

Solution: The point P corresponds to $t = 0$. Moreover,

$$\mathbf{r}'(t) =< 4,\ \cos t - 2\sin t,\ 2\cos t + \sin t > \quad \text{and} \quad \mathbf{r}'(0) =< 4, 1, 2 > .$$

$$\mathbf{r}''(t) =< 0,\ -\sin t - 2\cos t,\ -2\sin t + \cos t > \quad \text{and} \quad \mathbf{r}''(0) =< 0, -2, 1 > .$$

Therefore,

$$\kappa = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{|5\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}|}{21\sqrt{21}} = \frac{\sqrt{105}}{21\sqrt{21}} = \frac{\sqrt{5}}{21}.$$