

Review Sheet for Math 230, Midterm exam 2. Fall 2006

October 31, 2006

The second midterm exam will take place: Monday, November 13, from 8:15 to 9:30 pm. It will cover chapter 15 and sections 16.1 — 16.4, 16.6.

15.1. Functions of several variables. *Notions:* function of two variables, function of several variables, function of a single point variable, function of a single vector variable, domain of a function, range of a function, graph of a function of two variables, contour map, level curves, level surfaces.

Skills:

1. Evaluate a function of two or several variables at a point;
2. Find the domain of a function of two or several variables;
3. Find the range of a function of two or several variables;
4. Sketch the graph of a function of two variables (very roughly, in simple cases);
5. Sketch the (some) level curves of $f(x, y)$;
6. Sketch the (some) level surfaces of $f(x, y, z)$ (very roughly, in simple cases);
7. Interpret a contour map of $f(x, y)$ (a picture with some level curves).

15.2. Limits and continuity. *Notions:* limit of $f(x, y)$ as (x, y) approaches (a, b) , existence of limits, limits of elementary functions, continuity of $f(x, y)$ at (a, b) , continuity of the composition of two functions, polynomial function of two variables, rational function of two variables, continuity of polynomial functions and of rational functions (in their domain); and all that for function of three or more variables.

Skills: (mostly see 15.2.5–15.2.20, 15.2.27–15.2.34)

1. Find the limit of a function at a point, provided that it exists;
2. Show that the function has no limit at a given point;
3. Find out whether a function has a limit at a point;

4. Find the limit of a function at a point where it is continuous;
5. Find out whether a function is continuous at a point (applying the definition or constructing the function from elementary function by the means of the arithmetic operations and the composition;
6. Find the set of points where a given function is continuous.

15.3. Partial derivatives. *Notions:* partial derivative of $f(x, y)$ with respect to x (or y) at (a, b) or at a general point, same for functions of several variables, geometric interpretation of partial derivatives, second partial derivatives (four of them for $f(x, y)$, more for functions of several variables), higher derivatives, Clairaut's Theorem, its implication for the higher derivatives (p.952-953), partial differential equations.

Skills:

1. Find the first partial derivatives;
2. Find (all) the second partial derivatives;
3. Find the indicated (higher) partial derivatives (see 15.3.57–15.3.64), note that if you are asked to find f_{xy} , you should not find f_{yx} instead, unless you mention the Clairaut's theorem;
4. Apply the Clairaut's theorem, if it is applicable (it is not necessary, but simplifies the calculations);
5. Check whether a given function is a solution of a given partial differential equation (see 15.3.67–15.3.70);
6. Estimate partial derivatives (first and second) from a contour map of $f(x, y)$;
7. Determine the signs of the (first) partial derivatives for the function $f(x, y)$ by its graph or its contour map (see 15.3.5, 15.3.6, 15.3.66).

15.4. Tangent planes and linear approximations. *Notions:* tangent plane to the graph $z = f(x, y)$, the linearization (i.e. tangent plane approximation, i.e. $L(x, y)$) of $f(x, y)$ at (a, b) , a function differentiable at a point (definition), sufficient condition of being differentiable at a point (theorem), differentials, total differential, linearization of a function of several variables at a given point, linear approximation (i.e. $f(x, y) \approx L(x, y)$), the possibility of the absence of the tangent plane to the graph of a given function at a given point (see end of p.961).

Skills:

1. Find an equation of the tangent plane to the given surface at the specified point;
2. Find the linearization of a function of a point (or conclude it is not differentiable there) — see 15.4.11–15.4.16;

3. Find the (total) differential of a function (see 15.4.23–15.4.28);
4. Use differentials to estimate the maximum error in calculations (see 15.4.31–15.4.38).

15.5. The Chain Rule. *Notions:* the Chain Rule for functions of a single variable, all cases of the Chain Rule for functions of two or more variables, tree diagram, Implicit Function Theorem,.

1. Use the Chain Rule to find ordinary or partial derivatives (all cases) — see 15.5.1–15.5.12, 15.5.21–15.5.26;
2. Use the Chain Rule to solve problems about traveling particles, changing dimensions of parallelepipeds, cones, voltage in electrical circuits etc. (see 15.5.35–15.5.42);
3. Use implicit differentiation together with the Chain Rule (see examples 15.5.8, 15.5.9; see 15.5.27–15.5.34). It will not occur in the exam, but it is crucial for understanding more advanced Multivariable Calculus. Do not memorize the formulas 15.5.6, 15.5.7.

15.6. Directional Derivatives and the Gradient Vector. *Notions:* directional derivative of $f(x, y)$ at a given point in the direction of a given unit vector, same for functions of several variables, the gradient vector, connection between the gradient vector and directional derivatives, maximizing the directional derivative, tangent plane to the level surface $F(x, y, z) = k$ at a given point, normal line to the same surface at a given point, significance of the gradient vector.

Skills:

1. Show the direction of the gradient vector at a given point on the contour map (approximately) — see 15.6.36, 15.6.38;
2. Find the directional derivative at a given point in the direction of a given unit (!) vector or in the direction indicated by the angle θ ;
3. Find the directional derivative at a given point in the direction of a given nonunit vector (see 15.6.1–15.6.17);
4. Find the maximum rate of change of a function of two or more variables at a given point and the direction in which it occurs (see 15.6.21–15.6.26);
5. Find all points with the given direction of the fastest change of a given function — see 15.6.29;
6. Find the directions in which the directional derivative has a given value (such as 0 or -3) — see 15.6.28;
7. Find the tangent plane and the normal line to a given level surface $F(x, y, z) = k$ at a given point (see 15.6.39–15.6.44). If using symmetric equations to represent the normal line, avoid the division by zero;

8. See also 15.6.52–15.6.59.

15.7. Maximum and Minimum Values. *Notions:* local maximum/minimum of a function at a point, local maximum/minimum values of a function at a point, absolute maximum/minimum (value) of a function on its domain, critical point (same as stationary point), theorem (local minimum/maximum can be attained only at a critical point), extreme values (same as local minimum or maximum values), second derivative test, saddle point, example of a saddle point (hyperbolic paraboloid), closed set in R^2 , bounded set in R^2 , extreme value theorem for functions of two variables.

Skills:

1. Find the critical points of a function (including those where at least one of the partial derivatives does not exist);
2. Find the extreme values (local minima and maxima) and the saddle points of a function of two or more variables (see 15.7.5–15.7.18);
3. Use a contour map of a function to predict the location of the critical points of $f(x, y)$ and whether f has a saddle point or a local maximum or minimum at each of those points (see 15.7.3–15.7.4);
4. Classify a critical point using the Second Derivative Test;
5. Classify a critical point when the Second Derivative Test cannot be applied (simple cases);
6. Decide whether a set in R^2 is closed and bounded (simple cases);
7. Find the absolute minimum and maximum values of $f(x, y)$ on a given closed and bounded set (see 15.7.27–15.7.34);
8. Solve minimax problems (such as 15.7.37–15.7.50).

15.8. Lagrange Multipliers. *Notions:* the Method of Lagrange Multipliers with one or two constraints.

Skills:

1. Apply the Method of Lagrange Multipliers to find the minimum or maximum values of a function $f(x, y)$ (or $f(x, y, z)$) subject to one constraint of the form $g(x, y) = k$ (or $g(x, y, z) = k$);
2. Same with two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$ — only for functions of three variables;
3. Solve problems such as 15.8.3–15.8.17, 15.8.25–15.8.36.

16.1. Double Integrals over Rectangles. *Notions:* definite integral, geometric interpretation when $f(x) \geq 0$, double integral of $f(x, y)$ over the rectangle R , geometric interpretation when $f(x, y) \geq 0$, Riemann sum, double Riemann sum, Midpoint Rule for definite integrals, Midpoint Rule for double

integrals (as a method of approximation of the value of an integral), the average value of a single variable function on $[a, b]$, the average value of $f(x, y)$ on the rectangle R , linearity of the integral (double integral in particular).

Skills:

1. Use the Midpoint Rule to estimate a definite integral for a given n (number of subintervals);
2. Use the Midpoint Rule to estimate a double integral for given m, n (i.e. when R is divided into mn subrectangles) — see 16.1.1–16.1.6;
3. Roughly estimate the double integral or the average value of $f(x, y)$ on R using a contour map (see 16.1.8–16.1.10);
4. Evaluate a double integral by first identifying it as the volume of a solid (see 16.1.11–16.1.13).

16.2. Iterated Integrals. *Notions:* partial integration, iterated integral, Fubini's theorem.

Skills:

1. Perform partial integration with respect to x , $B(y) = \int_a^b f(x, y)dx$, and to y , $A(x) = \int_c^d f(x, y)dy$;
2. Calculate iterated integrals over rectangles (see 16.2.3–16.2.12);
3. Use Fubini's theorem to calculate double integrals over rectangles (see 16.2.13–16.2.20);
4. Reverse the order of integration over a rectangle (see example 16.2.3);
5. Interpret an iterated integral as the volume of a solid and sketch the solid (see 16.3.21–16.3.22);
6. Find volumes of various solids, not necessarily bounded by the xy -plane (see 16.2.23–16.2.29).

16.3. Double Integrals over General Regions. *Notions:* double integral over a plane bounded region D , two simple types of plane regions (referred in the textbook as 'type I' and 'type II', though it is not necessary to memorize the terms 'type I' and 'type II' — they are just for the sake of explanation).

Skills:

1. Find (any) rectangle R , enclosing a given plane region D ;
2. Evaluate double integrals over plane regions of type I;
3. Evaluate double integrals over plane regions of type II;
4. Find volumes of solids whose base is a plane region of type I or II (see 16.3.19–16.3.28);

5. All the same for more general kinds of regions, which can be represented as a union of regions of type I or type II (see 16.4.49, 16.4.50);
6. Change the order of integration in an integral over a rectangle or a region of type I or II (see example 16.3.5; 16.4.43–16.4.48).

16.4. Double Integrals in Polar Coordinates. *Notions:* polar coordinates, polar rectangle, Riemann sum in polar coordinates, double integral in polar coordinates, change to polar coordinates in a double integral.

Skills:

1. Perform the change to polar coordinates in a double integral (see p.1041);
2. Evaluate a double integral over a rectangle by changing to polar coordinates (see 16.4.9–16.4.16);
3. Evaluate a double integral over a general polar region of the form $D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ by changing to polar coordinates (see p.1042–1043);
4. Find the area of a general polar region as described above in polar coordinates (see example 16.4.3);
5. Use polar coordinates to find the volume of a given solid (see 16.4.21–16.4.27);
6. Interpret an iterated integral in polar coordinates as the volume of a solid;
7. Sketch the region whose area is given by the iterated integral in polar coordinates (or in rectangular coordinates) — see 16.4.7–16.4.8.

16.6. Surface area. *Notions:* surface area, using the tangent plane approximation to define the surface area.

Skills:

1. Find the surface area of some part of a given surface that lies *above* or *under* a given rectangle, triangle, disk, plane or below a given plane, or inside a given sphere, cylinder, paraboloid etc. — see example 16.6.1, 16.6.2; 16.6.1–16.6.12.