Math 230 **Fall 2004 Final**

#1

a.) $\perp$ b.) $\parallel$ c.) $\parallel$

#2

a.)

$$\nabla f = < y\cos(xy), x\cos(xy) + e^y >$$

$$\nabla f(2,0) = < 0, 2\cdot(1) + e^0 >$$

$$= < 0.3 >$$

b.) Want

$$D_{\vec{u}} f(2,0) = 0$$

$$= \nabla f \cdot \vec{u}$$

$$= < 0,3 > \cdot \vec{u} \qquad u = < a,b >$$

$$= 3b = 0$$

$$\to b = 0 \to < a,0 > \to < \pm 1,0 >$$

#3.

$$(x-1)^2 + (y-1)^2 + z^2 = 1$$
$$x - 1 = \sin\phi\cos\theta$$
$$y - 1 = \sin\phi\sin\theta$$
$$z = \cos\phi$$

$\rightarrow$

$$\boxed{\begin{array}{l} x = \sin\phi\sin\theta + 1 \\ y = \sin\phi\sin\theta + 1 \\ z = \cos\phi \\ 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}}$$

#4.

$$f_x = 2x = 0$$
$$x = 0$$

$$f_y = -3 + 3y^2 = 0$$
$$y^2 = 1$$
$$y = \pm 1$$

Critical Points

(0,1)

(0,-1)

$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= 12y$$

$$f_{xx} = 2$$
$$f_{yy} = 6y$$
$$f_{xy} = 0$$

$$D(0,1) = 12 > 0$$
$$D(0,-1) = -12 < 0$$

$$f_{xx} = 2 > 0$$

$\rightarrow$(0,1) local minimum

$\rightarrow$(0,-1) saddle point

7. EVALUATE THE INTEGRAL

$$\iint_S y \, dS$$

WHERE S IS THE PLANE $y+3-z=0$ THAT LIES IN THE CYLINDER $x^2+y^2=1$.

$\boxed{\text{SOLN}}$ Notice we can write our surface as

$$z = g(x,y) = y+3.$$

Thus,

$$\iint_S y \, dS = \iint_D y \cdot \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \; dA$$

$$= \iint_D y \cdot \sqrt{0+1+1} \; dA$$

$$= \sqrt{2} \iint_D y \; dA$$

Notice that our $D = \{ (r,\theta) \mid 0 \leq r \leq 1 , 0 \leq \theta \leq 2\pi \}$

So

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r^2 \sin\theta \, dr \, d\theta$$

$$= 0$$

9. LET $\vec{F}(x,y) = \langle 2x+\sin y,\ x\cos y \rangle$

a) SHOW CONSERVATIVE

Need to show $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = \cos y \quad , \quad \frac{\partial Q}{\partial x} = \cos y$$

Thus it is conservative.

b) Evaluate $\int_C F \cdot dr$ where C goes from $(0,1)$ to $(1,0)$.

We use the Fund. Thm. of Line Integrals. So we need to find a potential function for F. We first integrate our first component of F with respect to x,

$$f(x,y) = \int (2x+\sin y)\,dx = x^2 + x\sin y + g(y).$$

We then take the partial derivative w/ respect to y,

$$f_y(x,y) = x\cos y + g_y(y).$$

Comparing with the 2^{nd} component of

F, we see that $g_y(y) = 0$. Thus,

$$g(y) = \int g_y(y)\,dy = C \quad \text{(constant)}$$

So $\quad f(x,y) = x^2 + x\sin y + C$

So
$$\int_C F \cdot dr = \cancel{f(0,1)}$$
$$= f(1,0) - f(0,1)$$
$$= 1.$$

10) Use the divergence thm to ~~show~~ find the inward flux of the vector field

$$\vec{F}(x,y,z) = \langle yz, y, z \rangle$$

across the sphere $x^2 + y^2 + z^2 = 4$

$\boxed{\text{Soln}}$ Recall that our thm is used to show that

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

when our surface is oriented <u>OUTWARD</u>.

So for this problem we have

$$\cancel{\iint_S \langle yz, y, z \rangle}$$

$$\iint_S F \cdot dS = -\iiint_E \operatorname{div} F \, dV$$

We first calculate the div F,

$$\operatorname{div} F = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 2$$

So $$\iint_S F \cdot dS = -\iiint_E (2)\, dV$$

$$= -2 \iiint_E 1 \, dV$$

But integrating ~~sphere over~~ 1 over a sphere is just calculating the volume. Thus,

$$\iint_S F \cdot dS = -2 \iiint_E 1 \, dV = (-2)\left(\frac{4}{3}\pi (2)^3\right)$$

$$= -64\pi/3$$

11.

a) Find curl F

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & x-2y \end{vmatrix}$$

$$= \langle -2, -1, 2 \rangle$$

b) Use Stoke's thrm ....

Using Stoke's thm we see that

$$\iint_S \text{curl } F \cdot dS = \int_C F \cdot dr$$

where S is our paraboloid and C is the boundary of S. So C is when $z=0$, thus

$$0 = 1 - x^2 - y^2 \implies \text{circle of radius } 1.$$

To ensure C is positively oriented we want our circle to be going counter clockwise in direction. Thus

$$r(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le 2\pi$$

And $$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t, \cos t - 2\sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

12. Recall that the ~~normal~~ normal vector to the ~~plane~~ surface is given by

$$r_u \times r_v$$

Thus

$$r_u = \langle 1, \cos v, \sin v \rangle$$
$$r_v = \langle 0, -u\sin v, u\cos v \rangle$$

So $r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix}$

$$= \langle u, -u\cos v, -u\sin v \rangle$$

Note that the point $P(1,0,1)$ corresponds to $u=1$ and $v = \frac{\pi}{2}$. So

$r_u \times r_v$ at $u=1, v=\frac{\pi}{2}$ is $\langle 1, 0, -1 \rangle$

Thus the equation of the plane is

$$(1)(x-1) + (0)(y-0) + (-1)(z-1) = 0$$

$$x - z = 0$$