

Final exam. Math 230. Spring2005

December 13, 2006

Problem 1 (10 points) Suppose the acceleration of a particle in space is given by

$$\vec{a}(t) = \langle \sin(t), \cos(t), 6t \rangle.$$

Suppose the particle starts at rest, i.e., $\vec{v}(0) = \langle 0, 0, 0 \rangle$, with initial position given by $\langle 1, 2, 3 \rangle$.

1. (4 points) Find the velocity $\vec{v}(t)$ at any time t ;
2. (4 points) Find the position $\vec{r}(t)$ at any time t ;
3. (2 points) Find the speed of the particle after π seconds.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(\tau) d\tau = \langle 0, 0, 0 \rangle + \int_0^t \langle \sin(\tau), \cos(\tau), 6\tau \rangle d\tau = \langle \int_0^t \sin(\tau) d\tau, \int_0^t \cos(\tau) d\tau, \int_0^t 6\tau d\tau \rangle = \langle -\cos(t) + 1, \sin(t), 3t^2 \rangle$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau) d\tau = \langle 1, 2, 3 \rangle + \int_0^t \langle -\cos(\tau) + 1, \sin(\tau), 3\tau^2 \rangle d\tau = \langle 1, 2, 3 \rangle + \langle \int_0^t (-\cos(\tau) + 1) d\tau, \int_0^t \sin(\tau) d\tau, \int_0^t 3\tau^2 d\tau \rangle = \langle 1, 2, 3 \rangle + \langle -\sin(t) + t, -\cos(t) + 1, t^3 \rangle = \langle -\sin(t) + t + 1, -\cos(t) + 3, t^3 + 3 \rangle$$

$$|\vec{v}(\pi)| = | \langle -\cos(\pi) + 1, \sin(\pi), 3\pi^2 \rangle | = | \langle 2, 0, 3\pi^2 \rangle | = \sqrt{(2)^2 + (0)^2 + (3\pi^2)^2} = \sqrt{4 + 9\pi^4}$$

Problem 2 (10 points) The curve consists of two smooth pieces: the first, C_1 , is a straight line segment connecting the points $(3, -1)$ and $(1, 1)$ and the second, C_2 , follows the parabolic path $y = x^2$ from $(1, 1)$ to $(2, 4)$.

1. (3 points) Parameterize the path C_1 , including bounds on the parameter;
2. (2 points) Parameterize the path C_2 , including bounds on the parameter;
3. 5 points Compute $\int_C (x + 2y) dx + \sin(y) dy$.

Parameterize C_1 : $x = 3 - t, y = -1 - t, 0 \leq t \leq 2$. (Of course, there are many possible parameterizations).

Parameterize C_2 : $x = t, y = t^2, 1 \leq t \leq 2$.

$$\int_C (x + 2y) dx + \sin(y) dy = (\int_{C_1} (x + 2y) dx + \int_{C_1} \sin(y) dy) + (\int_{C_2} (x + 2y) dx + \int_{C_2} \sin(y) dy) = (\int_0^2 ((3 - t) + 2(-1 - t))(3 - t)' dt + \int_0^2 \sin(-1 - t)(-1 - t)' dt) + (\int_1^2 (t + 2t^2)(t)' dt + \int_1^2 \sin(t^2)(t^2)' dt) = \int_0^2 ((1 - 3t)(-1) + \sin(-1 - t)(-1)) dt +$$

$$\begin{aligned} \int_1^2 ((t + 2t^2)(1) + \sin(t^2)(2t))dt &= \int_0^2 (3t - 1 - \sin(-1 - t))dt + \int_1^2 (t + 2t^2 + \\ 2t \sin(t^2))dt &= (3t^2/2 - t - \cos(-1 - t))\Big|_0^2 + (t^2/2 + 2t^3/3 - \cos(t^2))\Big|_1^2 = ((6 - \\ 2 - \cos(-3)) - (-\cos(-1))) &+ ((4/2 + 16/3 - \cos(4)) - (1/2 + 2/3 - \cos(1))) = \\ 61/6 + \cos(-1) - \cos(-3) - \cos(4) &+ \cos(1) \\ (\cos(1) = \cos(-1), \text{ of course}) \end{aligned}$$

Problem 3 (15 points) Let E be the set

$$E = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + 4z^2 \leq 12\}.$$

Let $f(x, y, z) = xyz$. Find the absolute maximum and minimum values of $f(x, y, z)$ in E .

Find the critical points $f_x = f_y = f_z = 0$ yields $yz = xz = xy = 0$, which is in turn equivalent to " $x = y = 0$ or $x = z = 0$ or $y = z = 0$ ". The value of $f(x, y, z) = 0$ at all these points is zero. Clearly, 0 is neither the absolute maximum nor minimum value (try, for instance, $(1, 1, 1)$ and $(-1, 1, 1)$ to ensure).

Therefore, the absolute maximum and minimum values are attained somewhere on the boundary of the E (the surface of the ellipsoid) E . The first one is only sketched here briefly. It involves making a change of variables $w = 2z$ to convert the ellipsoid to a sphere. Without loss of generality, we can consider only $z \geq 0$ (since $f(x, y, z) = f(-x, y, -z)$, so the absolute max/min values are also attained in the upper hemisphere) and so parameterize the upper hemisphere. Then the problem amounts to finding min/max values of a function on a circle, which can be done in polar coordinates (by setting the derivatives with respect to r and θ to be equal to zero).

Another solution uses the method of Lagrange Multipliers. Unfortunately, the method cannot be applied directly as we studied it, because there is only one constraint in this case, so the function is to be maximized or minimized on a surface rather than on a curve. However, at the point where the absolute maximum or minimum is achieved, the gradient of f is still going to be orthogonal to the surface of the ellipsoid, and therefore collinear with and proportional to the gradient vector of $g(x, y, z) = x^2 + y^2 + 4z^2$, giving the direction of the normal line at this point. $\nabla g(x, y, z) = \langle 2x, 2y, 8z \rangle \neq 0$ (it is nonzero on the surface of the ellipsoid). $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$. Then $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, i.e.

$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 8z \rangle$. If $x = 0$ or $y = 0$ or $z = 0$, then $f(x, y, z) = 0$, which is neither the absolute minimum nor the absolute maximum, as discussed above. Therefore assume $x, y, z \neq 0$.

$yz = \lambda 2x$, hence $yz/x = 2\lambda = xz/y = xy/(4z)$. So that $yz/x = xz/y$, therefore $x^2 = y^2$; $yz/x = xy/(4z)$, therefore $4z^2 = x^2$. Assume now that $x, y, z \geq 0$. In this case $x = y = 2z$, and with the condition $x^2 + y^2 + 4z^2 = 12$, it gives the only point $(2, 2, 1)$. If we now abandon the assumption $x, y, z \geq 0$, we end up with 8 points: $(\pm 2, 2, 1), (\pm 2, -2, 1), (\pm 2, 2, -1), (\pm 2, -2, -1)$.

The absolute maximum value: $f(2, 2, 1) = f(-2, -2, 1) = f(-2, 2, -1) = f(2, -2, -1) = 4$.

The absolute minimum value: $f(-2, 2, 1) = f(2, -2, 1) = f(2, 2, -1) = f(-2, -2, -1) = -4$. me six candidate points as above.

Problem 4 (10 points) Find parametric equations for the line of intersection of the planes P_1 and P_2 :

$$P_1 : x + y + z = 2 \text{ and } P_2 : y - 3z + 6 = 0.$$

The normal vector to the first plane is $\vec{n}_1 = \langle 1, 1, 1 \rangle$. (Indeed, viewing the plane as a level surface, it is also the gradient vector at any point). The normal vector to the second plane is $\vec{n}_2 = \langle 0, 1, -3 \rangle$. Taking their cross product, $\vec{n} = \vec{n}_1 \times \vec{n}_2 = \langle -4, 3, 1 \rangle$. As far as $\vec{n} \neq \vec{0}$, the planes are not parallel, so there is one line which is their intersection. \vec{n} serves as the direction vector on this line. Now it suffices to find any point in the intersection of the planes to determine the line completely.

It means, find any triple (x, y, z) such that $x + y + z = 2$ and $y - 3z + 6 = 0$. As far as there are three variables and only two equations here, let us make an arbitrary decision: let $z = 0$. Then $x + y = 2$ and $y = -6$, so that $x = 8$. Thus the point $(8, -6, 0)$ lies on both planes. The parametric equations for the line of intersection of the planes are therefore

$$x = 8 - 4t, y = -6 + 3t, z = t \quad (-\infty \leq t \leq +\infty).$$

Problem 5 (10 points) Consider the function

$$f(x, y, z) = \ln(1 + xyz) + e^z \cos(xy)$$

$$\text{where } x = u \cos(v), y = u \sin(v), z = u^2.$$

Find the value of $\frac{\partial f}{\partial u}$ when $u = 2$ and $v = 0$.

When $u = 2$ and $v = 0$, $x(2, 0) = 2 \cos(0) = 2$, $y(2, 0) = 2 \sin(0) = 0$, $z = 2^2 = 4$. Apply the Chain Rule to simplify the computations. $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$. $\frac{\partial x}{\partial u} = \cos v$, $\frac{\partial y}{\partial u} = \sin v$, $\frac{\partial z}{\partial u} = 2u$. In particular, $\frac{\partial x}{\partial u} \Big|_{u=2, v=0} = \cos(0) = 1$, $\frac{\partial y}{\partial u} \Big|_{u=2, v=0} = \sin(0) = 0$, $\frac{\partial z}{\partial u} \Big|_{u=2, v=0} = 2 \cdot 2 = 4$.

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{yz}{1+xyz} - ye^z \sin(xy),$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{xz}{1+xyz} - xe^z \sin(xy),$$

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{xy}{1+xyz} + e^z \cos(xy).$$

When $u = 2$ and $v = 0$, $\frac{\partial f}{\partial x} \Big|_{u=2, v=0} = \frac{\partial f}{\partial x}(x(u, v), y(u, v), z(u, v)) \Big|_{u=2, v=0} = \frac{\partial f}{\partial x}(2, 0, 4) = \frac{0 \cdot 4}{1+2 \cdot 0 \cdot 4} - 0 \cdot e^4 \sin(2 \cdot 0) = 0$.

Similarly, $\frac{\partial f}{\partial y}(2, 0, 4) = \frac{2 \cdot 4}{1+2 \cdot 0 \cdot 4} - 2 \cdot e^4 \sin(2 \cdot 0) = 8 - 0 = 8$ and $\frac{\partial f}{\partial z}(2, 0, 4) = \frac{2 \cdot 0}{1+2 \cdot 0 \cdot 4} + e^4 \cos(2 \cdot 0) = 0 + e^4 \cdot \cos(0) = e^4$.

Finally, $\frac{\partial f}{\partial u} \Big|_{u=2, v=0} = \frac{\partial f}{\partial x}(x(u, v), y(u, v), z(u, v)) \frac{\partial x}{\partial u}(u, v) + \frac{\partial f}{\partial y}(x(u, v), y(u, v), z(u, v)) \frac{\partial y}{\partial u}(u, v) + \frac{\partial f}{\partial z}(x(u, v), y(u, v), z(u, v)) \frac{\partial z}{\partial u}(u, v) \Big|_{u=2, v=0} = (0) \cdot (1) + (8) \cdot (0) + (e^4) \cdot (4) = 4e^4$.

A straightforward solution without the Chain rule would be to evaluate $\frac{\partial}{\partial u}(\ln(1 + (u \cos(v))(u \sin(v))(u^2)) + e^{u^2} \cos((u \cos(v))(u \sin(v)))) \Big|_{u=2, v=0}$

Problem 6 (5 points) Suppose that $\iint_D f(x, y) dA = 16$ where D is the triangular region on xy -plane with vertices $(1, 0)$, $(3, 0)$ and $(2, 2)$. Find the average value of the function f over the region D .

This problem is very simple, it just test whether you remember the definition of the average value of a function over some region. By the definition, the average value of the function f over the region D is $f_{ave} = \frac{\iint_D f(x,y)dA}{A(D)}$, where $A(D)$ is the area of D . In this case, $A(D) = \frac{2 \cdot 2}{2} = 2$, therefore $f_{ave} = \frac{16}{2} = 8$.

Problem 7 (15 points) Let S be the boundary of the region of space E in the first octant, bounded above by $z = 4 - \sqrt{x^2 + y^2}$. Let S be oriented corresponding to outward facing normal vectors. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = (\sin(z) + e^x)\vec{i} + (1 - ye^x)\vec{j} + (3z - \cos(x^2))\vec{k}$.

It is important to understand the problem statement.
Pause for a little and describe the region E .

It is crucial that you understand, which region is meant. Otherwise you won't solve the problem.

Hint: $z = 4 - \sqrt{x^2 + y^2}$ **does not** describe a sphere or a part of a sphere. (Squaring both sides yields $z^2 = 16 + x^2 + y^2 - 8\sqrt{x^2 + y^2}$). For simplicity, start from $z = -\sqrt{x^2 + y^2}$. It is the lower half of the cone $z^2 = x^2 + y^2$. Therefore, $z = 4 - \sqrt{x^2 + y^2}$ is the same lower half of the same cone, translated 4 units along the z -axis. Finally, we keep only the part in the first octant.

There is yet another trick in this problem. What is S , the boundary of E ?
Pause for a little and describe the surface S .

Hint: S is **not just** the part of the half-cone $z = 4 - \sqrt{x^2 + y^2}$ that lies in the first octant. S is the boundary of E . The boundary of E consists of four smooth pieces, three of them lie in the three coordinate planes. It is too time-consuming to evaluate the integral in question directly.

Recognize that what is asked is to find the outward flow of the vector field F (i.e. the surface integral of the normal component of F) through the surface S . The Divergence Theorem can be applied. (The component functions of \vec{F} have continuous partial derivatives everywhere). Then $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} dV$ by the Divergence Theorem.

$\text{div} \vec{F} = \frac{\partial \sin(z) + e^x}{\partial x} + \frac{\partial 1 - ye^x}{\partial y} + \frac{\partial 3z - \cos(x^2)}{\partial z} = e^x - e^x + 3 = 3$. Thus, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 3dV = 3V(E)$, three times the volume of E . The volume of E can be found as $1/3 \cdot (\text{Area of the Base}) \cdot (\text{Height}) = 1/3 \cdot (\frac{1}{4}\pi 4^2) \cdot 4 = 16\pi/3$ or by direct integration, $V(E) = \int_0^{\pi/2} \int_0^4 z(r, \theta) r dr d\theta = \int_0^{\pi/2} \int_0^4 (4 - r) r dr d\theta = \int_0^{\pi/2} (2r^2 - r^3/3)|_0^4 d\theta = \int_0^{\pi/2} (32 - 64/3) d\theta = \frac{\pi}{2} (96/3 - 64/3) = \frac{\pi}{2} \frac{32}{3} = 16\pi/3$.

Therefore, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} dV = \iiint_E 3dV = 3V(E) = 3(16\pi/3) = 16\pi$.

Problem 8 1. (2 points) Let $\vec{F} = \langle P, Q \rangle$ and let C be a piecewise smooth curve bounding a closed region D of the plane. Explain what it means for C to be positively oriented.

2. (3 points) State Green's theorem for the situation described in part (a).
3. (10 points) Compute $\oint_C \vec{F} \cdot d\vec{r}$ where C is the positively oriented boundary of the region D bounded above by $y = 4 - x^2$ and below by the x -axis, where $\vec{F}(x, y) = \langle -y^2/2 - \cos(y), (y^2 - 2)^{1/3} + x \sin(y) \rangle$.
- Note: there was a typo in the problem statement, $+\cos(y)$ instead of $-\cos(y)$.

A simple closed curve is positively oriented if, as t increases, it is traversed in counterclockwise direction.

Let C be a positively oriented, piecewise-smooth, piecewise closed curve in the plane. Let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then $\int_C P dx + Q dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$.

C consists of two smooth pieces, C_1 and C_2 . Parameterize the first smooth piece, C_1 : $x = t, y = 0, -2 \leq t \leq 2$. Parameterize the second smooth piece, C_2 : $x = 2 - t, y = 4 - (2 - t)^2, 0 \leq t \leq 4$. Note that it would have been wrong to parameterize the second piece as $x = t, y = 4 - t^2, -2 \leq t \leq 2$, because the orientation should be positive.

Apply the Green's theorem. $P(x, y) = -y^2/2 - \cos(y)$, $Q(x, y) = (y^2 - 2)^{1/3} + x \sin(y)$, $F(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j}$. In fact, it is false that Q has continuous partial derivatives on an open region containing D (on the contrary, if $y = \sqrt{2}$, then $\frac{\partial Q}{\partial y}$ does not exist). This is, probably, a mistake in the problem; however, it turns out that the Green's Theorem still can be applied in this particular case.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \iint_D ((\sin(y)) - (-y + \sin(y))) dA = \iint_D (y) dA = \\ &= \int_{-2}^2 \int_0^{4-x^2} (y) dy dx = \int_{-2}^2 y^2/2 \Big|_{y=0}^{y=4-x^2} dx = \int_{-2}^2 \frac{(4-x^2)^2}{2} dx = \int_{-2}^2 \frac{16-8x^2+x^4}{2} dx = \\ &= \frac{16x-8x^3/3+x^5/5}{2} \Big|_{-2}^2 = \frac{32-64/3+32/5}{2} - \frac{-32+64/3-32/5}{2} = \frac{64-128/3+64/5}{2} = 32 \cdot (1 - \\ &2/3 + 1/5) = 32 \cdot \frac{15-10+3}{15} = 32 \cdot \frac{8}{15} = \frac{256}{15}. \end{aligned}$$

Problem 9 (15 points) 1. Using curl, show that $\vec{F}(x, y, z) = \langle x^2 + ye^x, e^x - y \cos(z), 2zx + \frac{1}{2}y^2 \sin(z) \rangle$ is conservative.

2. Find a potential function for \vec{F} .
3. Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is given by $\langle 4 \cos^3 t, 4 \sin^3(t), t \rangle$ where t varies from 0 to π .

$$\begin{aligned} \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 + ye^x & e^x - y \cos(z) & 2zx + \frac{1}{2}y^2 \sin(z) \end{vmatrix} = (\frac{\partial}{\partial y} (2zx + \\ &\frac{1}{2}y^2 \sin(z)) - \frac{\partial}{\partial z} (e^x - y \cos(z))) \vec{i} + (\frac{\partial}{\partial z} (z^2 + ye^x) - \frac{\partial}{\partial x} (2zx + \frac{1}{2}y^2 \sin(z))) \vec{j} + \\ &(\frac{\partial}{\partial x} (e^x - y \cos(z)) - \frac{\partial}{\partial y} (z^2 + ye^x)) \vec{k} = (y \sin(z) - y \sin(z)) \vec{i} + (2z - 2z) \vec{j} + \\ &(e^x - e^x) \vec{k} = \vec{0}. \vec{F} \text{ is defined on } \mathbf{R}^3 \text{ and } \text{curl } F = \vec{0}. \text{ (Also, its component} \end{aligned}$$

functions, $z^2 + ye^x$, $e^x - y \cos(z)$ and $2zx + \frac{1}{2}y^2 \sin(z)$ have continuous partial derivatives). Therefore, F is conservative.

By definition, a potential function for F , $g(x, y, z)$ must satisfy $\nabla g(x, y, z) = \vec{F}(x, y, z)$. That is,
$$\begin{cases} g_x(x, y, z) = z^2 + ye^x \\ g_y(x, y, z) = e^x - y \cos(z) \\ g_z(x, y, z) = 2zx + \frac{1}{2}y^2 \sin(z) \end{cases}$$

Start from any of the three conditions. For instance, $g_y(x, y, z) = e^x - y \cos(z)$. It implies that $g(x, y, z) = ye^x - \frac{y^2}{2} \cos(z) + h(x, z)$ (how otherwise can it be the case that $g_y = e^x - y \cos(z)$?). In fact, this is just (partial) integration: $g(x, y, z) = \int g_y(x, y, z) dy + h(x, z) = \int (e^x - y \cos(z)) dy + h(x, z) = ye^x - \frac{y^2}{2} \cos(z) + h(x, z)$.

Substitute this result into the other two conditions. $g_x(x, y, z) = z^2 + ye^x$, but $g_x(x, y, z) = \frac{\partial}{\partial x}(ye^x - \frac{y^2}{2} \cos(z) + h(x, z)) = ye^x - 0 + \frac{\partial h(x, z)}{\partial x}$. Similarly, $g_z(x, y, z) = 2zx + \frac{1}{2}y^2 \sin(z)$, but $g_z(x, y, z) = \frac{\partial}{\partial z}(ye^x - \frac{y^2}{2} \cos(z) + h(x, z)) = \frac{y^2}{2} \sin(z) + \frac{\partial h(x, z)}{\partial z}$.

Comparing these results, we conclude that $\frac{\partial h(x, z)}{\partial x} = z^2$ and $\frac{\partial h(x, z)}{\partial z} = 2zx$. Applying the same method to find $h(x, z)$ (since we know $\nabla h(x, z) = \langle z^2, 2zx \rangle$), $h(x, z) = xz^2 + p(z)$ and, substituting, $\frac{\partial h(x, z)}{\partial z} = 2xz + p'(z) = 2zx$, so that $p'(z) = 0$, $p(z) = C$ (any constant) and $h(x, z) = xz^2 + C$. Finally, $g(x, y, z) = ye^x - \frac{y^2}{2} \cos(z) + h(x, z) = ye^x - \frac{y^2}{2} \cos(z) + xz^2 + C$ works for any constant C .

In particular, taking, say, $C = 0$, $\nabla(ye^x - \frac{y^2}{2} \cos(z) + xz^2) = \langle ye^x + z^2, e^x - y \cos(z), \frac{y^2}{2} \sin(z) + 2zx \rangle = \vec{F}(x, y, z)$, as required.

C is a smooth curve and F is conservative. Thus, $\int_C \vec{F} \cdot d\vec{r} = g(\vec{r}(\pi)) - g(\vec{r}(0))$, where $g(x, y, z)$ is a potential function for F and the curve C is given by $\vec{r}(t) = \langle 4 \cos^3 t, 4 \sin^3 t, t \rangle$, $0 \leq t \leq \pi$.

Then $\vec{r}(\pi) = \langle 4 \cos^3(\pi), 4 \sin^3(\pi), \pi \rangle = \langle -4, 0, \pi \rangle$ and $\vec{r}(0) = \langle 4 \cos^3(0), 4 \sin^3(0), 0 \rangle = \langle 4, 0, 0 \rangle$. Hence, $\int_C \vec{F} \cdot d\vec{r} = g(-4, 0, \pi) - g(4, 0, 0)$. Taking $g(x, y, z) = ye^x - \frac{y^2}{2} \cos(z) + xz^2$ found above, $g(-4, 0, \pi) = 0e^{-4} - \frac{0^2}{2} \cos(\pi) + (-4)\pi^2 = -4\pi^2$ and $g(4, 0, 0) = 0e^4 - \frac{0^2}{2} \cos(0) + 4 \cdot 0^2 = 0$.

Finally, $\int_C \vec{F} \cdot d\vec{r} = (-4\pi^2) - (0) = -4\pi^2$.

Problem 10 (10 points) Let $\vec{F} = \vec{F}(x, y, z)$ denote a vector field, and $f = f(x, y, z)$ denote a scalar function. Determine whether each of the expressions yields a vector field, a scalar field or has no meaning. Each part is worth 1 point.

1. $\text{curl}(\nabla f)$ is a vector field, since ∇f is a vector field.
2. $\text{div}(\text{div} \vec{F})$ has no meaning, since $\text{div} \vec{F}$ is already a scalar field, but the (outer) div should be applied to a vector field.
3. $\nabla(\nabla \times \vec{F})$ is a vector field, the curl of the curl of \vec{F} .

4. $\vec{F} \cdot (\nabla \times \vec{F})$ is a scalar field, the dot product of two vector fields.
5. $\text{curl}(\text{curl}f)$ is meaningless, since the curl of a scalar field f is undefined.
6. $\text{div}(\text{curlgrad}f)$ is a scalar field, the divergence of a vector field $\text{curlgrad}f$.
7. $\nabla(\vec{F} \cdot \nabla f)$ is a vector field, the gradient of a scalar field, which is the dot product of two vector fields, \vec{F} and ∇f .
8. $\text{curl}(f) \times \text{div}(\vec{F})$ is meaningless, since $\text{curl}(f)$ is undefined.
9. $\nabla(\text{div}\vec{F})$ is a vector field, the gradient of a scalar field $\text{div}\vec{F}$.
10. $(\nabla \times \vec{F}) \cdot \nabla f$ is a scalar field as the dot product of two vector fields.

Problem 11 (15 points) Using Stokes' Theorem, evaluate the outward flux of $\text{curl}\vec{F}$, i.e. $\iint_S \text{curl}\vec{F} \cdot \vec{dS}$ where $\vec{F} = \langle x, y, y \rangle$ and S is the non-closed surface consisting of the part of the cylinder $x^2 + y^2 = 4$ in $0 \leq z \leq 4$ and the part of the plane $z = 4$ inside the cylinder $x^2 + y^2 = 4$.

The Stokes' Theorem makes this problem extremely simple. The most difficult part is to imagine (or sketch) the surface S and to find out, what is the boundary curve (if there is only one). In fact, the boundary curve C is just the circle $x^2 + y^2 = 4$ in the xy -plane. C can be parameterized as $x = 2\cos(t), y = 2\sin(t), z = 0, 0 \leq t \leq 2\pi$. This orientation is positive, because it agrees with the orientation of the surface. (Provided it is clear from the problem statement, what is the 'outward flux', because, strictly speaking, the surface S is not closed).

Then, by the Stokes' Theorem, $\int_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\vec{F} \cdot \vec{dS}$, and this is what we want to find. It is simple to evaluate $\int_C \vec{F} \cdot \vec{dr}$. In fact, \vec{F} is always orthogonal to the curve C . Pick a point on the curve, $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$. The tangent vector at this point is $\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 0 \rangle$, and $\vec{r}'(t) \cdot \vec{F}(2\cos(t), 2\sin(t), 0) = \langle -2\sin(t), 2\cos(t), 0 \rangle \cdot \langle 2\cos(t), 2\sin(t), 0 \rangle = 0$, using the fact that $\vec{F}(x, y, z) = \langle x, y, y \rangle$, as given.

Thus, $0 = \int_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\vec{F} \cdot \vec{dS}$.

Another way to get this answer is to observe that $\text{curl}\vec{F} = \langle 1, 0, 0 \rangle = \vec{i}$. The flow of $\text{curl}\vec{F}$ through the circle $x^2 + y^2 \leq 4, z = 0$ (i.e. the part of the plane $z = 0$ inside the cylinder $x^2 + y^2 = 4$) is therefore zero (because $\text{curl}\vec{F}$ is parallel to the xy -plane). Hence, we can, without loss of generality, add to S the bottom part (the circle, mentioned above) — the flow through this side is zero, so it won't change the answer. But now we need to find the outward flux of the $\text{curl}\vec{F}$ through a closed surface, which is zero, because $\text{curl}\vec{F}$ is incompressible (and by the Divergence Theorem).

Problem 12 (15 points) Evaluate the line integral $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F} = \langle e^{-x} + \tan(x), x^2, e^{z^2} + \cosh(z) \rangle$ and C is the boundary of the part of the plane

$2x + 2y + z = 10$ in the first octant, oriented counterclockwise when viewed from above.

(Note that $\cosh(z) = \frac{e^z + e^{-z}}{2}$, not to be confused with $\cos(z)$).

The mentioned part of the plane is obviously a triangle. What are the coordinates of its vertices? Find the intersections of the plane $2x + 2y + z = 10$ with the coordinate axes.

x -axis: substitute $y = z = 0$, obtain $2x = 10$, or $x = 5$. y -axis: substitute $x = z = 0$, obtain $2y = 10$, or $y = 5$. z -axis: substitute $x = y = 0$, obtain $z = 10$. Therefore, the three vertices of the triangle are $(5, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 10)$.

C is piecewise smooth, it consists of the three sides of the triangle. $C = C_1 + C_2 + C_3$. C_1 goes from $(5, 0, 0)$ to $(0, 5, 0)$, C_2 goes from $(0, 5, 0)$ to $(0, 0, 10)$ and C_3 goes from $(0, 0, 10)$ to $(5, 0, 0)$. Then C is indeed oriented counterclockwise when viewed from above.

Parameterize:

$$C_1: x = 5 - 5t, y = 5t, z = 0, 0 \leq t \leq 1.$$

$$C_2: x = 0, y = 5 - 5t, z = 10t, 0 \leq t \leq 1.$$

$$C_3: x = 5t, y = 0, z = -10t, 0 \leq t \leq 1.$$

$$\text{Then } \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} = ?$$

Here a significant simplification can be achieved. Represent $F(x, y, z) = \langle e^{-x} + \tan(x), x^2, e^{z^2} + \cosh(z) \rangle$ as $F(x, y, z) = F_1(x, y, z) + F_2(x, y, z)$, where $F_1(x, y, z) = \langle e^{-x} + \tan(x), 0, e^{z^2} + \cosh(z) \rangle$ and $F_2(x, y, z) = \langle 0, x^2, 0 \rangle$.

Then $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}_1 \cdot d\vec{r} + \int_C \vec{F}_2 \cdot d\vec{r}$. But F_1 is conservative (easy to see that $\text{curl} \vec{F}_1 = \vec{0}$) and C is closed; therefore, $\int_C \vec{F}_1 \cdot d\vec{r} = 0$ and $\int_C \vec{F} \cdot d\vec{r} = 0 + \int_C \vec{F}_2 \cdot d\vec{r} = \int_{C_1} \vec{F}_2 \cdot d\vec{r} + \int_{C_2} \vec{F}_2 \cdot d\vec{r} + \int_{C_3} \vec{F}_2 \cdot d\vec{r}$.

$\vec{F}_2 = x^2 \vec{j}$, so now a direct evaluation is possible. Using the parametrization for C_1 , $\int_{C_1} \vec{F}_2 \cdot d\vec{r} = \int_{C_1} 0 dx + x^2 dy + 0 dz = \int_{C_1} x^2 dy = \int_0^1 (5 - 5t)^2 y'(t) dt = \int_0^1 (5 - 5t)^2 (5) dt = \left. \frac{(5-5t)^3}{-3} \right|_0^1 = 125/3$.

In fact, $\int_{C_2} \vec{F}_2 \cdot d\vec{r} = \int_{C_2} x^2 dy = \int_0^1 (0)^2 (5 - 5t)' dt = 0$, indeed, C_2 lies in the yz -coordinate plane, i.e. $x = 0$, so that F_2 is zero for all points on C_2 . Also, $\int_{C_3} \vec{F}_2 \cdot d\vec{r} = \int_{C_3} x^2 dy = \int_0^1 (5t)^2 (0)' dt = 0$. Since C_3 lies in the xz -coordinate plane, i.e. $y = 0$, integrating with respect to y can give only zero. In terms of vector fields, $\int_{C_3} \vec{F}_2 \cdot d\vec{r} = 0$, because F_2 is orthogonal to the xz -coordinate plane, containing the curve C_3 , the tangential component of F_2 along the curve C_3 is, therefore, always zero.

Therefore, $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}_2 \cdot d\vec{r} = \int_{C_1} \vec{F}_2 \cdot d\vec{r} + \int_{C_2} \vec{F}_2 \cdot d\vec{r} + \int_{C_3} \vec{F}_2 \cdot d\vec{r} = 125/3 + 0 + 0 = 125/3$.

Problem 13 (5 points) Convert the triple integral $\iiint_E (x+z) dV$ where E is the solid region in xyz -space that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside of the sphere $x^2 + y^2 + z^2 = 9$, into spherical coordinates. DO NOT EVALUATE THIS INTEGRAL.

E can be viewed as a spherical wedge. It is described by two conditions: $\phi \leq \pi/4$ (above the cone $\phi = \pi/4$) and $\rho \leq 3$ (inside the sphere). So that $E = \{(\rho, \theta, \phi) | 0 \leq \rho \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4\}$.

Using $x = (\rho \sin(\phi)) \cos(\theta)$, $z = \rho \cos(\phi)$, we can rewrite the integral as $\iiint_E (x + z) dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^3 ((\rho \sin(\phi)) \cos(\theta) + \rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\theta d\phi$.