

## Answers to Christmas Quiz 13

Math 230. Friday, 12/15/6

There were two problems, both graded out of 4 points. The best score was then chosen. One extra point was then added in any case, so the result was from 1 to 5.

**Problem 1 (4 points)** Given a vector field  $\vec{F} = \langle z, x, x \rangle$ . Find any closed curve  $C$  such that  $\int_C \vec{F} \cdot d\vec{r} = 1$  (or  $-1$ ).

The idea is, of course, to apply the Stokes' Theorem. The Stokes' Theorem claims that under certain assumptions, which will be true (since we are going to find some curve that is easy to describe, and so, probably, is piecewise smooth). The Stokes' Theorem then asserts that  $\int_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl } \vec{F} \cdot d\vec{S}$ , where  $D$  is the region enclosed by  $C$ .

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & x \end{vmatrix} = (\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(x)) \vec{i} + (\frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}(x)) \vec{j} + (\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)) \vec{k} = \langle 0, 0, 1 \rangle = \vec{k}.$$
 Then it is natural to look for a flat curve lying completely in the  $xy$ -plane (or in any horizontal plane), since we are given the flux of  $\vec{k}$  through the region enclosed by this curve.  $\text{curl } \vec{F} = \vec{k}$  is constant throughout any horizontal plane; thus, if  $C$  is any flat curve lying in any horizontal plane and  $D$  is enclosed by  $C$  and oriented upwards, then  $\iint_D \text{curl } \vec{F} \cdot d\vec{S} = \iint_D \vec{k} \cdot d\vec{S} = A(D)$ , the area of  $D$ . In order for  $D$  to be oriented upwards (so as to satisfy the conditions of the Stokes' Theorem),  $C$  has to be coherently oriented, i.e. counterclockwise when viewed from above.

So in order to obtain  $\int_C \vec{F} \cdot d\vec{r} = 1$ , one may take a circle with area  $\pi r^2 = 1$ , hence its radius  $r = 1/\sqrt{\pi}$ . The circle has to be oriented counterclockwise. The simplest one is, of course, the one centered at the origin. The standard parametrization is then  $x = \frac{\cos(t)}{\sqrt{\pi}}, y = \frac{\sin(t)}{\sqrt{\pi}}, z = 0, 0 \leq t \leq 2\pi$ .

If we need  $\int_C \vec{F} \cdot d\vec{r} = -1$ , then we need to choose the other orientation (negative when viewed from above), so that  $D$  in the Stokes' Theorem ends up being oriented downwards, while  $\text{curl } \vec{F}$ , so that its curl through  $D$  is negative. The immediate answer is then  $x = \frac{\cos(t)}{\sqrt{\pi}}, y = \frac{\sin(-t)}{\sqrt{\pi}}, z = 0, 0 \leq t \leq 2\pi$ .

One minor terminological issue: strictly (and technically) speaking, one should not say that one possible answer is a hemisphere or that the orientation is upwards or downwards. Since we were asked to provide a curve  $C$ , it can

be a circle or something else, but not a hemisphere, which is a surface. Also, the orientation of a curve is usually described as clockwise or counterclockwise when viewed from, say, above, but not upwards or downwards. (Of course, one can explicate: the orientation of the curve  $C$  agrees with the upward orientation of the surface  $D$ , enclosed by  $C$ ). d.

Note that the question was to find a closed curve. It is immediate to find any curve  $C$  satisfying the rest of the criteria. For instance, take the curve  $C$  to be the one given by the parametric equations  $x = t, y = 0, z = 1, 0 \leq t \leq 1000$ . Then  $\int_C \vec{F} \cdot d\vec{r} = \int_C (zdx + xdy + xdz) = \int_C zdx = 1 \cdot \int_C dx = 1000$ .

**Problem 2 (4 points)** Let  $S$  be the unit sphere  $x^2 + y^2 + z^2 = 1$  oriented outwards. Find any vector field  $\vec{F}(x, y, z)$  such that  $\iint_S \vec{F} \cdot d\vec{S} = 1$  (or  $\pi$ ).

This can be easily answered using the Divergence Theorem. We will try to find some nice  $\vec{F}$ , of course, so that we have no problems applying the theorem. (We shall not find  $\vec{F}$  with non-differentiable component functions). Let  $E$  stand for the unit ball  $x^2 + y^2 + z^2 \leq 1$ . Clearly,  $S$  is the boundary of  $E$ . Then, by the Divergence Theorem,  $\iiint_E \operatorname{div} \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$ , i.e. the outward flux of  $\vec{F}$  through the surface is equal to the triple integral of the divergence of  $\vec{F}$  over the region enclosed by  $S$ .

The required value of the flux is given. How can we guarantee a certain triple integral of the divergence of  $\vec{F}$ ? The simplest way is to assume that the divergence of  $\vec{F}$  is a constant,  $\operatorname{div} \vec{F} = c$ . Then  $\iiint_E \operatorname{div} \vec{F} dV = c \cdot \iiint_E dV = c \cdot V(E)$ ,  $c$  times the volume of  $E$ .  $V(E) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$ . Hence,  $\iiint_E \operatorname{div} \vec{F} dV = c \frac{4\pi}{3}$ .

If we need  $\iiint_E \operatorname{div} \vec{F} dV = \pi$ , then just find  $c$  such that  $c \frac{4\pi}{3} = \pi$ . Clearly,  $c = 3/4$  works. Take any vector field with divergence  $3/4$ , such as  $\vec{F}(x, y, z) = \langle \frac{3x}{4}, 0, 0 \rangle$  or  $\vec{G}(x, y, z) = \frac{1}{4} \langle x, y, z \rangle$ . Similarly to obtain  $\iiint_E \operatorname{div} \vec{F} dV = 1$  take, for instance,  $\vec{H}(x, y, z) = \langle \frac{3x}{4\pi}, 0, 0 \rangle$ .

Note that in general if  $\operatorname{div} \vec{F}$  is not a constant then, of course,

$\iiint_E \operatorname{div} \vec{F} dV \neq \operatorname{div} \vec{F}(?, ?, ?) \cdot \iiint_E dV$ , because it is not clear, at which point should we evaluate the divergence in the expression  $\operatorname{div} \vec{F}(?, ?, ?)$  (though there usually would be *some* point, possibly unknown to us, making it true).

Also if you don't remember the formula for the volume of a sphere of radius  $r$ , it is reasonably simple to derive it by integrating in spherical coordinates. If you remember instead that the surface of a sphere is  $4\pi r^2$ , then it suffices to notice that  $V(r)' = S(r)$ , i.e.  $(\frac{4}{3}\pi(r)^3)' = 4\pi r^2$ .