Math 251 Spring 2004 Midterm Exam I

1. Both $f(t,y) = \dfrac{1}{y-3t^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{-1}{(y-3t^2)^2}$ have to be continuous. The only condition needed is that $y_0 - 3t_0^2 \neq 0$

2. a) The equilibrium solutions are $y = -3, 0, 1$.

b) $y = -3$ is unstable, $y = 0$ is (asymptotically) stable, and $y = 1$ is unstable.

c) $\lim\limits_{t\to\infty} y(t) = \infty$

d) $\lim\limits_{t\to\infty} y(t) = 0$

3. a) $M(x,y) = y^2 e^{xy^2} + 4x^3$, $N(x,y) = 2xye^{xy^2} + 2$

$$\frac{\partial M}{\partial y} = 2xy^3 e^{xy^2} + 2ye^{xy^2} = \frac{\partial N}{\partial x}$$

So the equation is an exact equation.

b) $e^{xy^2} + x^4 + 2y = 5$

4. a) The roots of the characteristic equation $r^2 - 4r - 5 = 0$ are $r = -1, 5$. Therefore, the functions $y_1 = e^{-t}$ and $y_2 = e^{5t}$ form a pair of fundamental solutions for this second order linear differential equation.

b) General solution is $y(t) = C_1 y_1 + C_2 y_2 = C_1 e^{-t} + C_2 e^{5t}$.

c) $y(t) = \frac{7}{6}e^{-t} + \frac{5}{6}e^{5t}$

5. a) (Solved as a separable equation.) $e^y = \sin t + t + e^3$ (implicit solution), or $y = \ln(\sin t + t + e^3)$ (explicit solution).

b) (Solved as a first order linear equation.) $y(t) = \dfrac{2\ln|t|}{t} + \dfrac{2}{t}$.

6. a) The roots of the characteristic equation are $r = -1 \pm \sqrt{2}i$, the general solution is therefore $y(t) = C_1 e^{-t}\cos\sqrt{2}t + C_2 e^{-t}\sin\sqrt{2}t$.

b) The root of the characteristic equation is $r = -3$ (repeated), the general solution is therefore $y(t) = C_1 e^{-3t} + C_2 te^{-3t}$.

7. a) non-linear; b) linear; c) non-linear; d) non-linear; e) linear.

8. a) yes; b) no; c) no.

9. First identify that $p(t) = \dfrac{-4}{t}$; and $t_0 = 1$ so the solution interval is $(0,\infty)$.

By Abel's Theorem, $W(y_1,y_2) = Ce^{-\int p(t)\,dt} = Ce^{\int \frac{4}{t}\,dt} = Ce^{4\ln t} = Ct^4$.

Calculate the Wronskian determinant $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ at $t = 1$, $W(y_1(1), y_2(1)) = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$, and equating it with the result obtained by Abel's Theorem: $3 = C(1^4)$, hence $C = 3$. Therefore, $W(y_1,y_2) = 3t^4$.

10. $Q(t) = 120\gamma - 120\gamma e^{\frac{-1}{60}t}$. The limiting amount of salt is $\lim_{t\to\infty} Q(t) = 120\gamma\,(grams)$. (The limiting concentration would be γ $grams/liter$, but that is not what the question asks.)