Math 251 – Sections 1/2
November 7, 2005 Second Exam

ANSWER KEY

There are 9 questions on this exam. Question 1 is worth 20 points. Questions 2 through 9 are worth 10 points each. The total number of points is 100. If a question has multiple parts, then the points assigned to the question are divided equally among the parts, unless otherwise indicated.

Where appropriate, **show your work** to receive credit; partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone.

Time limit 1 hour and 15 minutes.

1. **i.** One of the following ODE's represents the displacement $y(t)$ in a spring-mass system with resonance? Circle it.

$3y''+108y = 4\sin 6t$ $\quad$ $3y''+108 = 2\sin 6t$ $\quad$ $3y''+108y = 6\cos 3t$ $\quad$ $3y''+111y'+108y = \cos 6t$

ANS. Frequency of external force and natural frequency match in the last equation. So its the first one.

ii For a spring-mass system system with mass equal to 4 kg, spring contant equal to 9 N/m, which damping constant γ causes critical damping?

ANS. $\gamma = \sqrt{4mk} = 6$

iii For a spring-mass system system with mass equal to 4 kg, spring contant equal to 9 N/m, damping constant equal to 5, and external force equal to $\sin t$ what does the transient part of the solution look like? **(Do not determine any constants.)**

ANS. Since $\gamma^2 - 4mk$ is negative the transient part of the solution is a combination of sine and cosine multiplied by a decaying exponential.

iv. What is the **definition** of the Laplace transform of $g(t)$?

ANS.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

v What is the following Laplace transform: $\mathcal{L}\{t^2 e^{2t}\}$

ANS. $\dfrac{2}{(s-2)^3}$

vi. Circle all the functions among the following **that have a Laplace transform**:

$$e^{t^3} \qquad \frac{t}{1-t} \qquad \frac{|t-1|}{t-1} \qquad e^{t^{1/2}}$$

ANS. The last two.

vii. Suppose that the Laplace transform of y is Y. If $y(0) = 2$ and $y'(0) = -3$, then find the Laplace transform of y''.

ANS. $s(sY - 2) + 3$

viii. Suppose that a force of $-3\delta(t-1)$ acts on an object of mass 1 kg. If $y(0) = 2$ and $y'(0) = 2$, then find $y'(2)$.

ANS. $y'(2) = -1$.

ix Suppose that the homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ has only one eigenvalue $r_1 = 3$ and that all eigenvectors are multiples of a single vector. Find the solution satifying the IVP: $\mathbf{x}(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

ANS. $e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

x Which of the following matrices could be the matrix of the linearization at the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the system of ODE's for a damped pendulum ? _________B

And which could be the one at the critical point $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$? _________F

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

2. In parts **a** and **b** determine the form of a particular solution y_p having the **least** number of unknown constants. **DO NOT DETERMINE** the unknown constants appearing in your answers in parts **a** and **b**.

a. 2pt $y'' - 12y' + 36y = (2t^2 - 1)e^t$

ANS. $(At^2 + Bt + C)e^t$

b. 2pt $y'' - 15y' + 36y = (3t^2 + 2t)e^{3t}$

ANS. $t(At^2 + Bt + C)e^t$

c. 6pt Without using Laplace transforms, find a particular solution to the following ODE:

$$y'' + 2y' + 9y = 2\sin 3t$$

(In this part you need to **determine the unknown constant(s) in the solution.**

ANS. We find a particular solution of the complexified equation $y'' + 2y' + 9y = 2e^{3it}$ and take its imaginary part. We guess that $y_C = Ae^{3it}$ since $3i$ is not a root of the characteristic polynomial. Plugging in gives $L[y_C] = A(-9 + 2(3i) + 9)e^{3it} = 2e^{3it}$. We find that $A = 1/3i = -i/3$ The imaginary part of $(-i/3)e^{3it}$ is $y_p = (-1/3)\cos(3t)$.

3. An object with mass 3 kg stretches a spring 10/9 meters to its equilibrium position. Assume that there is no damping device attached. Also assume that at time $t = 0$ the object is released 3 meter below its equilibrium position with a upward velocity of 12 meter/sec.

a. 2pt Write down a differential equation for $y(t)$ with initial conditions for the displacement of the object from its equilibrium position?

ANS. $m = 3, \gamma = 0, k = 27$. The ODE is : $y'' + 9y = 0$ and $y(0) = 3, y'(0) = -12$.

b. 2pt Find a formula for $y(t)$

ANS. $y = 3\cos(3t) - 4\sin(3t)$.

c. 2pt Find the maximum value of $y(t)$.

ANS. 5

d. 2pt If a damping device with damping constant γ is added to this system, then explain why a periodic external force cannot cause resonance regardless of the period.

ANS. Need $Ae^{i\omega t}$ to be a solution to the homogeneous equation for resonance. However, a purely imaginary number $i\omega$ is not a root of a quadratic $mr^2 + m\gamma + k$, with $\gamma \neq 0$ by the quadratic formula.

4. **a. 4pt** Find the function $f(t)$ whose Laplace transform is equal to $\dfrac{s}{s^2+4s+40}$

ANS. $e^{-2t}\cos 6t + \frac{2}{6}e^{-2t}\sin 6t.$

b. 4pt Find the function $f(t)$ whose Laplace transform is equal to $\dfrac{s}{s^2+4s+3}$

ANS. $e^{-2t}\cosh t + 2e^{-2t}\sinh t.$

5. **a.** Consider the function

$$f(t) = \begin{cases} t, & \text{if} \quad t < 1 \\ 2-t, & \text{if} \quad 1 \le t < 2 \\ 0, & \text{if} \quad 2 \le t \end{cases}$$

Sketch a graph of this function and find a formula for $f(t)$ in terms of unit step functions $u(t-c)$, for appropriate values of c.

ANS. $t(u(t) - u(t-1)) + (2-t)(u(t-1) - u(t-2)) = tu(t) - (2t-2)u(t-1) + (t-2)u(t-2)$

b. Find the Laplace transform of the function $\sin t u(t - \pi/2)$

ANS. $\cos(t - \pi/2) = \sin t$ So the Laplace transform is: $e^{-\pi/2} \dfrac{s}{s^2+1}$

6. Solve the following IVP:

$$y' + 2y = tu(t-2) + \delta(t-2) \qquad y(0) = 1,$$

ANS. Taking the Laplace transform of both sides gies

$$sY - 1 + 2Y = \frac{e^{-2s}}{s^2} + 2\frac{e^{-2s}}{s} + e^{-2s}$$

Hence, solving for Y

$$\begin{aligned} Y &= \frac{1}{s+2} + \frac{e^{-2s}}{s+2} + \frac{e^{-2s}}{(s+2)s^2} + 2\frac{e^{-2s}}{(s+2)s)}e^{-2s} \\ &= \frac{1}{s+2} + e^{-2s}\left(\frac{3}{4(s+2)} + \frac{1}{4s} + \frac{1}{2s^2}\right) \\ &= \mathcal{L}\{e^{-2t} + u(t-2)\left(\frac{3}{4}e^{-2(t-2)} + \frac{1}{4} + \frac{1}{2}(t-2)\right)\} \end{aligned}$$

7. In each part of this Problem do the following:
i. Sketch a phase portrait for this system.
ii. State the name associated with the critical point at $(0,0)$ and state whether it is stable, asymptotically stable or unstable?

a. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. Since the eigenvalues are negatives, the origin is an asymptotically stable node. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

b. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. Since the eigenvalues have opposite sign, the origin is an unstable saddle. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

c. 2pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS. There is only one eigenvalue and every vector is an eigenvector. The origin is an unstable proper node. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

8. **a. 6pt** Find the general solution of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 3 & -2 \\ 2 & -3 \end{pmatrix}$.

ANS. The characteristic polynomial is $(r-3)^2+4$. A complex eigenvalue is: $3+2i$. Then $A-(3+2i)I = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}$ and hence the corresponding eigenvector is $\begin{pmatrix} -1 \\ i \end{pmatrix}$. So a complex solution is

$$e^{(3+2i)t}\begin{pmatrix} -1 \\ i \end{pmatrix} = e^{3t}(\cos 2t + i\sin 2t)\begin{pmatrix} -1 \\ i \end{pmatrix}$$

We obtain real solutions by taking the real and imaginary parts of the above:

$$\mathbf{x}_1 = e^{3t}\begin{pmatrix} -\cos 2t \\ -\sin 2t \end{pmatrix} \qquad \mathbf{x}_2 = e^{3t}\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

b. 2pt Sketch a phase portrait for the system given in Part a.

c. 1pt What is the critical point called?

d. 1pt What is its stability?

ANS. The origin is an unstable spiral for this system. Also, since $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ the spiral is in the clockwise direction.

To see the phase portrait use the phase portrait applet.

9. Consider the nonlinear system of first order differential equations:

$$\begin{aligned} x' &= x(2-y) \\ y' &= y(x-4) \end{aligned}$$

a. 1pt Find the critical point(s) of this system.

ANS. Setting the two right hand sides equal to 0 we find two critical points $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

b. 5pt Approximate the nonlinear system with a linear system near each critical point found in part **a** ,

ANS. At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is $\begin{matrix} x' = 2x \\ y = -4y \end{matrix}$ which has eigenvalues $r_1 = 2$ and $r_2 = -4$ which have opposite sign. Hence the origin is a saddle for for this system. Moveover the trajectory along the positive real x axis moves away from the origin and the one along the y axis moves towards the origin.
At $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ we set $\begin{matrix} u = x - 4 \\ v = y - 2 \end{matrix}$. Substituting this into the original equation gives

$$\begin{aligned} u' &= (u+4)(-v) \approx -4v \\ v' &= (v+2)u \approx 2u \end{aligned}$$

The eigenvalues of this system are purely imaginary. Thus this critical point is a center for the linear system. Moreover the trajectories are traversed clockwise since A applied to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. 2pt State the name and stability of the approximating linear system at each critical point. Also state whether or not it is likely that the stability of the approximating linear system reflects that of the nonlinear system.

ANS. saddle, unstable, likely to reflect the original system center, stable, not likely to reflect the origina system but sometimes we get lucky.

d. 2pt Sketch a phase portrait for the original system.

ANS. To see the phase portrait use the phase portrait applet.