

Math 251
March 31, 2005 Second Exam

NAME: _____ Section #: _____

There are 9 questions on this exam. Question 9 is worth 12 points. Each other question is worth 11 points. The points assigned to each part of the question are indicated at the start of the part.

Show all your work. Partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone before starting this exam.

Time limit 1 hour and 15 minutes.

Question	Score
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

1. In Parts **a** and **b** determine the form of a particular solution $y_p = y_p(t)$ having the **least** number of unknown constants. **DO NOT DETERMINE** the unknown constants appearing in your answers in Parts **a** and **b**.

a. **2pt** $y'' - 14y' + 49y = 2t^2e^{7t}$

b. **2pt** $y'' - 50y' + 49y = 3te^t$

- c. **7pt** Without using Laplace transforms, find a particular solution to the following ODE:

$$y'' + 3y = e^t \sin 2t$$

(In this part you need to **determine the unknown constant(s) in the solution.**)

2. Assume that acceleration due to gravity g is equal to 10 meter/sec². An object with mass 2 kg stretches a spring 2.5 meters to the equilibrium position. Assume that there is no damping device attached and also assume that at time $t = 0$ the object is released 1 meter below its equilibrium position with a upward velocity of 4 meter/sec.

a. **3pt** Write down a differential equation with initial conditions for $y(t)$ for the displacement of the object below its equilibrium position.

b. **4pt** Find a formula for $y(t)$

c. **2pt** Find the maximum value of $y(t)$.

d. **2pt** If a periodic external force equal to $3 \cos \omega t$ Newtons is applied, then for what positive value of ω does resonance occur?

3. **a. 4pts** For a spring-mass system with mass equal to 1 kg, spring constant equal to 25 Newton/meter, which damping constant γ causes critical damping?

b. 3pts If the damping constant γ in the above system is set to 2 Newton-sec/meter, then what can be said about the number of times does the object pass through its equilibrium position?

c. 4pts If the damping constant γ in the above system is set to 8 Newton-sec/meter, then what is the interval of time between the second time the object returns to its equilibrium position and the third time it returns to its equilibrium position?

4. **a. 2pts** What is the definition of the Laplace transform $\mathcal{L}\{e^{3t}\}$?

b. 4pts Use the answer to Part **a** to calculate $\mathcal{L}\{e^{3t}\}$. (Be sure to explain why this exists only when $s > 3$).

c. 2pts Suppose that the Laplace transform of y is Y . If $y(0) = 2$ and $y'(0) = -3$, then find the Laplace transform of y'' .

d. 3pts Find the function $f(t)$ whose Laplace transform is equal to $\frac{s}{s^2 + 2s + 5}$

5. a. 5pt $\frac{e^{-3s}}{(s-1)(s+3)}$

b. 6pt Assume that acceleration due to gravity g is equal to 10 meter/sec². An object with mass 2 kg stretches a spring 4 m to equilibrium. At time $t = 0$ it is released 2 meters below its equilibrium position with an upward velocity of 3 meters/sec. At time $t = 6$ it is struck with a hammer and as a result its momentum is **decreased** by 7 kg-meters/sec at that moment in time. At time $t = 8$ a constant external force of 9 Newtons is added and at $t = 10$ it is removed. Write down a differential equation with initial conditions for the displacement $y(t)$ of the object below its equilibrium position. **DO NOT SOLVE THIS EQUATION**

6. a. 3pt Consider the function

$$f(t) = \begin{cases} t, & \text{if } t < 2 \\ 2, & \text{if } 2 \leq t \end{cases}$$

Sketch a graph of this function and find a formula for $f(t)$ in terms of unit step functions $u(t - c)$, for appropriate values of c .

b. 4pt Determine the Laplace transform of $f(t)$ in Part **a**

c. 3pt Find the Laplace transform of $u(t - \pi) \sin(t)$.

7. 11pt Solve the following IVP:

$$y' + 3y = \delta(t - 1) + u(t - 2)$$

$$y(0) = -4$$

8. a. **7pt** Find the general solution of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. (Note: the solution should be real.)

b. **2pt** Sketch a phase portrait for the system given in Part a.

c. **1pt** Which of the six names for the critical points fits the critical point of this system.

d. **1pt** What is its stability?

9. In Parts **a** and **b** of this Problem do the following:
- Sketch a phase portrait for this system.
 - State the name associated with the critical point at $(0,0)$ and state whether it is stable, asymptotically stable or unstable?

a. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:

$$\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

b. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:

$$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

In parts **c.** and **d.** of this Problem do the following:
determine the stability and the name associated with the critical point of the system at the origin.

c. 2pt $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

d. 2pt $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.