

ECON 4323/5301 Homework 1 Solutions

Fall 2007

September 25, 2007

1. Let $k = 7$,

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.$$

Calculate $A + B$, $B - A$, kB , A^t , AB , and AB^t , if possible.

- (a) (5 points)

$$A + B = \begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{pmatrix},$$

- (b) (5 points)

$$B - A = \begin{pmatrix} -2 & -2 & -2 \\ 4 & 0 & 0 \end{pmatrix},$$

- (c) (5 points)

$$kB = \begin{pmatrix} 0 & 7 & -7 \\ 28 & -7 & 14 \end{pmatrix},$$

- (d) (5 points)

$$A^t = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{pmatrix},$$

- (e) (5 points) AB is undefined, and

- (f) (5 points)

$$AB^t = \begin{pmatrix} 2 & 7 \\ -3 & 5 \end{pmatrix}.$$

2. Find the determinant (5 points each) and the inverse (5 points each) of each of the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

$$\det(A) = 1, A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\det(B) = -12, B^{-1} = \begin{pmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},$$

$$\det(C) = -24, C^{-1} = \begin{pmatrix} 2 & 9/2 & -15/2 & 11/2 \\ 1/3 & -7/3 & 10/3 & -8/3 \\ -1/4 & 3/4 & -1 & 3/4 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

3. (5 points) Calculate the solution to the following system of linear equations:

$$\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$$

$$(x, y, z) = (1, -1, -2).$$

4. (5 points each) Determine the definiteness of the following symmetric matrices:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

The 1st and 2nd order leading principal minors are 2 and 1, respectively, so A is positive definite.

$$B = \begin{pmatrix} -3 & 4 \\ 4 & 4 \end{pmatrix}.$$

The 1st and 2nd order leading principal minors are -3 and -28, respectively, so B is indefinite.

$$C = \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}.$$

The 1st and 2nd order leading principal minors are -3 and 16, respectively, so C is negative definite.

$$D = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}.$$

The 1st, 2nd, and 3rd order leading principal minors are 3, 3, and 3, respectively, so D is positive definite.

$$E = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

The 1st, 2nd, and 3rd order leading principal minors are -3, 5, and -25, respectively, so E is negative definite.

$$F = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}.$$

The 1st, 2nd, and 3rd order leading principal minors are 1, 0, and -25, respectively, so F is indefinite.

5. Find the first (5 points) and second (5 points) derivatives of the following functions:

(a) $f(x) = x^7 + 3x^6 - 4x^2 + 5$

$$\begin{aligned} f'(x) &= 7x^6 + 18x^5 - 8x, \\ f''(x) &= 42x^5 + 90x^4 - 8. \end{aligned}$$

(b) $f(x) = (3x^3 - 1)(x^2 + 7)$

$$\begin{aligned} f'(x) &= (9x^2)(x^2 + 7) + (3x^3 - 1)(2x) = 15x^4 + 63x^2 - 2x, \\ f''(x) &= 60x^3 + 126x - 2. \end{aligned}$$

(c) $f(x) = (x^2 - 1)/(x^2 + 1)$

$$\begin{aligned} f'(x) &= \frac{(2x)(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}, \\ f''(x) &= \frac{4(x^2 + 1)^2 - (4x)(2(x^2 + 1)2x)}{(x^2 + 1)^4} = \frac{4 - 12x^2}{(x^2 + 1)^3} \end{aligned}$$

(d) $f(x) = (x^3 - 4x^2 + 1)^5$

$$\begin{aligned} f'(x) &= 5(x^3 - 4x^2 + 1)^4(3x^2 - 8x) = (15x^2 - 40x)(x^3 - 4x^2 + 1)^4, \\ f''(x) &= (30x - 40)(x^3 - 4x^2 + 1)^4 + (15x^2 - 40x)(4)(x^3 - 4x^2 + 1)^3(3x^2 - 8x). \end{aligned}$$

(e) $f(x) = 3x^{2/3} + 3x^{-1}$

$$\begin{aligned}f'(x) &= 2x^{-1/3} - 3x^{-2}, \\f''(x) &= -\frac{2}{3}x^{-4/3} + 6x^{-3}.\end{aligned}$$

(f) $f(x) = e^{x^2+3x}$

$$\begin{aligned}f'(x) &= (2x + 3)e^{x^2+3x}, \\f''(x) &= 2e^{x^2+3x} + (2x + 3)(2x + 3)e^{x^2+3x} = (2 + (2x + 3)^2)e^{x^2+3x}(2 + (2x + 3)^2).\end{aligned}$$

(g) $f(x) = \ln(x^2 + 4)$

$$\begin{aligned}f'(x) &= \frac{2x}{x^2 + 4}, \\f''(x) &= \frac{(2)(x^2 + 4) - (2x)(2x)}{(x^2 + 4)^2}.\end{aligned}$$

6. For the following functions, find the slope of the tangent line at $x = 5$ (5 points each) and determine the intervals where f is increasing (5 points each):

(a) $f(x) = 3x^2 - 12x + 13$

The slope of the line tangent to f at $x = 5$ is

$$f'(5) = f'(x)|_{x=5} = 6x - 12|_{x=5} = 6(5) - 12 = 18.$$

f is increasing for x such that $f'(x) > 0$. We can see that

$$f'(x) = 6x - 12 > 0 \Rightarrow x > 2.$$

Thus, f is increasing for $x \in (2, \infty)$.

(b) $f(x) = 2x/(x^2 + 2)$

The slope of the line tangent to f at $x = 5$ is

$$f'(5) = f'(x)|_{x=5} = \frac{(2)(x^2 + 2) - (2x)(2x)}{(x^2 + 2)^2}|_{x=5} = \frac{4 - 2x^2}{(x^2 + 2)^2}|_{x=5} = -\frac{46}{729}.$$

f is increasing for x such that $f'(x) > 0$. Since

$$f'(x) = \frac{4 - 2x^2}{(x^2 + 2)^2} > 0 \Rightarrow 4 - 2x^2 > 0 \Rightarrow 2 > x^2 \Rightarrow \sqrt{2} > x > -\sqrt{2},$$

it follows that f is increasing for $x \in (-\sqrt{2}, \sqrt{2})$.

7. (10 points) Let $f(x) = 10x^a$, where $a > 0$. For what values of a is f convex? Concave?

To simplify the problem, restrict the domain of f to \mathbf{R}_+ . We know that f is convex when $f''(x) \geq 0$ and that f is concave when $f''(x) \leq 0$. Since $f''(x) = 10a(a-1)x^{a-2}$, we can see that $f''(x) \geq 0$ if $a \geq 1$ and $f''(x) \leq 0$ if $a \leq 1$. Thus, f is convex if $a \geq 1$ and f is concave if $a \leq 1$.

8. Find all the first- and second- order partial derivatives of the following functions:

(a) (10 points) $f(x, y, z) = 3xyz + x^2y - xz^3$

$$f_x = 3yz + 2xy - z^3, f_y = 3xz + x^2, f_z = 3xy - 3xz^2,$$

$$f_{xx} = 2y, f_{xy} = f_{yx} = 3z + 2x, f_{xz} = f_{zx} = 3y - 3z^2,$$

$$f_{yy} = 0, f_{yz} = f_{zy} = 3x, f_{zz} = -6xz.$$

(b) (10 points) $f(x, y, z) = x^4/(yz)$

$$f_x = \frac{4x^3}{yz}, f_y = -\frac{x^4}{y^2z}, f_z = -\frac{x^4}{yz^2},$$

$$f_{xx} = \frac{12x^2}{yz}, f_{xy} = f_{yx} = -\frac{4x^3}{y^2z}, f_{xz} = f_{zx} = -\frac{4x^3}{yz^2},$$

$$f_{yy} = \frac{2x^4}{y^3z}, f_{yz} = f_{zy} = \frac{x^4}{y^2z^2}, f_{zz} = \frac{2x^4}{yz^3}.$$

(c) (not graded) $f(x, y, z) = (x^2 + y^3 + z^4)^6$

$$f_x = 6(x^2 + y^3 + z^4)^5(2x) = 12x(x^2 + y^3 + z^4)^5,$$

$$f_y = 6(x^2 + y^3 + z^4)^5(3y^2) = 18y^2(x^2 + y^3 + z^4)^5,$$

$$f_z = 6(x^2 + y^3 + z^4)^5(4z^3) = 24z^3(x^2 + y^3 + z^4)^5,$$

$$f_{xx} = 12(x^2 + y^3 + z^4)^5 + 12x(5)(x^2 + y^3 + z^4)^4(2x) \\ = 12(x^2 + y^3 + z^4)^5 + 120x^2(x^2 + y^3 + z^4)^4,$$

$$f_{xy} = f_{yx} = 60x(x^2 + y^3 + z^4)^4(3y^2) = 180xy^2(x^2 + y^3 + z^4)^4,$$

$$f_{xz} = f_{zx} = 60x(x^2 + y^3 + z^4)^4(4z^3) = 240xz^3(x^2 + y^3 + z^4)^4,$$

$$f_{yy} = 36y(x^2 + y^3 + z^4)^5 + 18y^2(5)(x^2 + y^3 + z^4)^4(3y^2) \\ = 36y(x^2 + y^3 + z^4)^5 + 270y^4(x^2 + y^3 + z^4)^4,$$

$$f_{yz} = f_{zy} = 90y^2(x^2 + y^3 + z^4)^4(4z^3) = 360y^2z^3(x^2 + y^3 + z^4)^4,$$

$$f_{zz} = 72z^2(x^2 + y^3 + z^4)^5 + 24z^3(5)(x^2 + y^3 + z^4)^4(4z^3) \\ = 72z^2(x^2 + y^3 + z^4)^5 + 480z^6(x^2 + y^3 + z^4)^4.$$

(d) (10 points) $f(x, y) = x^2 + e^{2y}$

$$f_x = 2x, f_y = 2e^{2y}, f_{xx} = 2, f_{xy} = f_{yx} = 0, f_{yy} = 4e^{2y}.$$

(e) (10 points) $f(x, y) = y \ln x$

$$f_x = y/x, f_y = \ln x, f_{xx} = -y/x^2, f_{xy} = f_{yx} = 1/x, f_{yy} = 0.$$

(f) (not graded) $f(x, y) = xy^2 - e^{xy}$

$$f_x = y^2 - ye^{xy}, f_y = 2xy - xe^{xy}, f_{xx} = -y^2 e^{xy}, f_{xy} = f_{yx} = 2y - e^{xy} - xy e^{xy}, f_{yy} = 2x - x^2 e^{xy}.$$