ECON 4323/5301 Fall 2007

Homework 2 Solutions

Grading:

Each of the following is worth 5 points:

1. Problem 1a
2. Problem 1b
3. Problem 1c
4. Problem 2a
    a. First order conditions
    b. Hessian matrix
5. Problem 2b – first order conditions
6. Problem 3
    a. Objective function
    b. First order conditions
7. Problem 4
    a. Lagrangian
    b. First order conditions
    c. Bordered Hessian
8. Problem 6
    a. Lagrangian
    b. First order conditions
    c. Bordered Hessian
9. Problem 8
    a. Parameterizing the problem correctly
    b. Using the Envelope Theorem correctly
10. Problem 10
11. Problem 11a
12. Problem 11b
13. Problem 11d

Problem 1.

a) $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3 = 0$; $\quad x_1 = 6, x_2 = 3$ implies $27 + y^3 = 0$. So, $y = -3$.

b) $(\partial F/\partial y)(6, 3, -3) = 27 \neq 0$; so the equation implicitly defines y as a function of x around the point $(6, 3, -3)$.

c) $$\frac{\partial y}{\partial x_1}(6,3) = -\frac{\partial F/\partial x_1}{\partial F/\partial y} = -\frac{2x_1}{3y^2} = -\frac{12}{27} = -\frac{4}{9} \quad \text{and}$$

$$\frac{\partial y}{\partial x_2}(6,3) = -\frac{\partial F/\partial x_2}{\partial F/\partial y} = \frac{2x_2}{3y^2} = \frac{6}{27} = \frac{2}{9}.$$

Problem 2.

a) $f(x, y, z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$. The first order conditions are

$$\frac{\partial f}{\partial x} = 2x + 6y - 10 = 0, \tag{1}$$

$$\frac{\partial f}{\partial y} = 6x + 2y - 3z - 5 = 0, \tag{2}$$

and

$$\frac{\partial f}{\partial z} = -3y + 8z - 21 = 0. \tag{3}$$

Equation (1) implies that $x = 5 - 3y$ and equation (3) implies that $z = 21/8 + (3/8)y$. Substituting for x and z in equation (2) and solving for y yields $y = 1$. It follows that $x = 2$ and $z = 3$.

The Hessian matrix is

$$H = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}. \tag{4}$$

Since the Hessian matrix is constant, $H(2, 1, 3)$ is given by (4). We can see that $\det(H_1) = \det(2) = 2 > 0$, and

$$\det(H_2) = \det \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} = 4 - 36 = -32 < 0.$$

This is enough to see that H is indefinite, so (2,1,3) is a saddle point.

$b)$ $f_x = 2x\, e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2x)\, e^{-(x^2+y^2+z^2)}$,
$f_y = 4y\, e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2y)\, e^{-(x^2+y^2+z^2)}$,
$f_z = 6z\, e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2z)\, e^{-(x^2+y^2+z^2)}$.
$f_x = 0$ implies $2x(x^2 + 2y^2 + 3z^2 - 1) = 0$.
$f_y = 0$ implies $2y(x^2 + 2y^2 + 3z^2 - 2) = 0$.
$f_z = 0$ implies $2z(x^2 + 2y^2 + 3z^2 - 3) = 0$.

Case 1: $x = y = z = 0$.

Case 2: $x^2 + 2y^2 + 3z^2 = 1$ and $y = z = 0$ implies $x = \pm 1$. So, $(1, 0, 0)$ and $(-1, 0, 0)$.

Case 3: $x^2 + 2y^2 + 3z^2 = 2$ and $x = z = 0$ implies $y = \pm 1$. So, $(0, 1, 0)$ and $(0, -1, 0)$.

Case 4: $x^2 + 2y^2 + 3z^2 = 3$ and $x = y = 0$ implies $z = \pm 1$. So, $(0, 0, 1)$ and $(0, 0, -1)$.

Hessian:

$$\mathrm{H} = \begin{pmatrix} 2A[(1-B) - 2x^2(2-B)] & -4xyA(3-B) & -4xzA(4-B) \\ -4xyA(3-B) & 2A[(2-B) - 2y^2(4-B)] & -4yzA(5-B) \\ -4xzA(4-B) & -4yzA(5-B) & 2A[(3-B) - 2z^2(6-B)] \end{pmatrix},$$

where $A = \exp(-(x^2 + y^2 + z^2))$ and $B = x^2 + 2y^2 + 3z^2$.

$$H(0,0,0) = A \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$ which is positive definite, so $(0, 0, 0)$ is a local min,

$$H(\pm 1,0,0) = e^{-1} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$ which is indefinite, so $(\pm 1, 0, 0)$ are saddles,

$$H(0,\pm 1,0) = e^{-1} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ which is indefinite, so $(0, \pm 1, 0)$ are saddles,

$$H(0,0,\pm 1) = e^{-1} \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \end{pmatrix}$$ which is negative definite, so $(0, 0, \pm 1)$ are local maxes.

Problem 3.

$$\text{Maximize } F(q_1, q_2) = q_1(10 - q_1) + q_2(16 - q_2) - (10 + (q_1 + q_2)^2)$$
$$= 10q_1 - q_1^2 + 16q_2 - q_2^2 - 10 - (q_1 + q_2)^2.$$

$$F_{q_1} = 10 - 2q_1 - 2(q_1 + q_2) = 0 \quad \text{or} \quad 4q_1 + 2q_2 = 10,$$
$$F_{q_2} = 16 - 2q_2 - 2(q_1 + q_2) = 0 \quad \text{or} \quad 2q_1 + 4q_2 = 16.$$

$q_2 = 11/3$ and $q_1 = 2/3$; $p_2 = 16 - 11/3 = 37/3$ and $p_1 = 10 - 2/3 = 28/3$.

$\Pi = (2/3)(28/3) + (11/3)(37/3) - 10 - (13/3)^2 = 204/9 = 68/3 = 22\frac{2}{3}$.

Problem 4.

Since $f(z) = z^2$ is a strictly increasing function, we can simplify the problem by maximizing $(\sqrt{x^2+y^2})^2 = x^2 + y^2$ rather than $\sqrt{x^2+y^2}$.

The problem is

$$\max \quad x^2 + y^2$$
$$\text{subject to} \quad x^2 + xy + y^2 = 3.$$

The Lagrangian is $L = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$. The first order conditions are

$$L_x = 2x - 2\lambda x - \lambda y = 0$$
$$L_y = 2y - \lambda x - 2\lambda y = 0$$
$$L_\lambda = -(x^2 + xy + y^2 - 3) = 0.$$

There are four solutions:

$$(x, y, \lambda) = \begin{cases} (-\sqrt{3}, \sqrt{3}, 2) \\ (\sqrt{3}, -\sqrt{3}, 2) \\ (1, 1, 2/3) \\ (-1, -1, 2/3) \end{cases}$$

The bordered Hessian is

$$H = \begin{bmatrix} 0 & 2x+y & x+2y \\ 2x+y & 2-2\lambda & -\lambda \\ x+2y & -\lambda & 2-2\lambda \end{bmatrix}. \qquad (5)$$

.

A critical point (x^*, y^*, λ^*) is a local maximum if $(-1)^k \det(H_k^B) > 0$ for all $k = m+1, \ldots, n$, where m is the number of constraints, n is the number of choice variables, and H_k^B is the matrix remaining after deleting all but the first $m + k$ rows and columns from H^B. A critical point is a local minimum if $(-1)^m \det(H_k^B) > 0$ for all$k = m+1, \ldots, n$. In this problem, there are 2 choice variables and 1 constraint, so it must be that $k=2$. Since $m + k = 1 + 2 = 3$, it follows that $H_k^B = H^B$. Evaluate the critical points one at a time:

1. $(x^*, y^*, \lambda^*) = (-\sqrt{3}, \sqrt{3}, 2)$. We can see that

$$H^B(-\sqrt{3}, \sqrt{3}, 2) = \begin{bmatrix} 0 & -\sqrt{3} & \sqrt{3} \\ -\sqrt{3} & -2 & -2 \\ \sqrt{3} & -2 & -2 \end{bmatrix} \tag{6}$$

.

It follows that

$$(-1)^2 \det(H_2^B(-\sqrt{3}, \sqrt{3}, 2)) = \det(H^B(-\sqrt{3}, \sqrt{3}, 2)) = 24 > 0,$$

so this critical point is a local maximum. Just to be careful, notice that

$$(-1)^1 \det(H_2^B(-\sqrt{3}, \sqrt{3}, 2)) = \det(H^B(-\sqrt{3}, \sqrt{3}, 2)) = -24 < 0,$$

so this critical point does not satisfy the criteria for a local minimum.

2. $(x^*, y^*, \lambda^*) = (\sqrt{3}, -\sqrt{3}, 2)$. Following the same evaluation procedure we used for the first critical point, we find that this critical point is also a local maximum.

3. $(x^*, y^*, \lambda^*) = (1, 1, 2/3)$. Following the same evaluation procedure we used for the first critical point, we find that this critical point is a local minimum.

4. $(x^*, y^*, \lambda^*) = (-1, -1, 2/3)$. Following the same evaluation procedure we used for the first critical point, we find that this critical point is a local minimum.

Problem 5.

The location and type of the critical points are independent of $k > 0$, so assume without loss of generality that $k = 1$.

$$\begin{aligned} \max \quad & x_1^a x_2^{1-a} \\ \text{subject to} \quad & (p_1 x_1 + p_2 x_2 - I) = 0. \end{aligned}$$

The Lagrangian is

$$L = x_1^a x_2^{1-a} - \lambda(p_1 x_1 + p_2 x_2 - I).$$

The first order conditions are

$$\begin{aligned} L_{x_1} &= a x_1^{a-1} x_2^{1-a} - \lambda p_1 = 0 \\ L_{x_2} &= (1-a) x_1^a x_2^{-a} - \lambda p_2 = 0 \\ L_\lambda &= p_1 x_1 + p_2 x_2 - I = 0. \end{aligned}$$

The solution is

$$x_1 = \frac{aI}{p_1} \qquad x_2 = \frac{(1-a)I}{p_2}.$$

The bordered Hessian is

$$H^B = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & a(a-1)x_1^{a-2}x_2^{1-a} & a(a-1)x_1^{a-1}x_2{-}a \\ p_2 & a(a-1)x_1^{a-1}x_2{-}a & -a(a-1)x_1^a x_2^{-a-1} \end{bmatrix}.$$

.

A critical point (x^*, y^*, λ^*) is a local maximum if $(-1)^k \det(H_k^B) > 0$ for all $k = m+1, \ldots, n$, where m is the number of constraints, n is the number of choice variables, and H_k^B is the matrix remaining after deleting all but the first $m + k$ rows and columns from H^B. In this problem, there are 2 choice variables and 1 constraint, so it must be that *k=2*. Since $m + k = 1 + 2 = 3$, it follows that $H_k^B = H^B$. We can see that

$$(-1)^2 \det(H^B(x^*, y^*, \lambda^*)) = 2p_1p_2a(1-a)x_1^{a-1}x_2^{-a} - p_2^2 a(a-1)x_1^{a-2}x_2^{1-a} + p_1^2 a(1-a)x_1^a x_2^{-a-1}.$$

Since prices are positive, x_1^* is positive, x_2^* is positive, and $0 < a < 1$, we can see that the first term is positive, the second term is negative, and the third term is positive. Adding together two positive terms and subtracting a negative term generates a positive term, so the result must be positive. Thus, the critical point must be a maximum.

Problem 6.

$$\begin{aligned} \min \quad & x^2 + y^2 + z^2 \\ \text{subject to} \quad & 3x + y + z = 5 \\ & x + y + z = 1. \end{aligned}$$

The Lagrangian is

$$L(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 - \lambda_1(3x + y + z - 5) - \lambda_2(x + y + z - 1).$$

The first order conditions are

$$L_x = 2x - 3\lambda_1 - \lambda_2 = 0$$

$$L_y = 2y - \lambda_1 - \lambda_2 = 0$$

$$L_z = 2z - \lambda_1 - \lambda_2 = 0$$

$$L_{\lambda_1} = 3x + y + z - 5 = 0$$

$$L_{\lambda_2} = x + y + z - 1 = 0.$$

This linear system of five equations in five unknowns has a unique solution:

$$(x^*, y^*, z^*, \lambda_1^*, \lambda_2^*) = (2, -1/2, -1/2, 5/2, -7/2).$$

The bordered Hessian is

$$H^B = \begin{bmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

A critical point is a local minimum if $(-1)^m \det(H_k^B) > 0$ for all$k = m + 1, \ldots, n$. In this problem, there are 3 choice variables and 2 constraints, so it must be that *k=3*. Since

$m + k = 2 + 3 = 5$, it follows that $H_k^B = H^B$. A little calculation shows that $(-1)^2 \det(H^B) = 16 > 0$, so the critical point is a minimum.

Problem 7.

The Lagrangian is

$$L(x, y, z, \lambda, \mu) = xz + yz - \lambda(y^2 + z^2 - 1) - \mu(xz - 3).$$

The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= z - \mu z = 0, & (1)\\
\frac{\partial L}{\partial y} &= z - 2\lambda y = 0, & (2)\\
\frac{\partial L}{\partial z} &= x + y - 2\lambda z - \mu x = 0, & (3)\\
\frac{\partial L}{\partial \lambda} &= -(y^2 + z^2 - 1) = 0, & (4)\\
\frac{\partial L}{\partial \mu} &= -(xz - 3) = 0. & (5)
\end{aligned}$$

It follows from a little algebra that there are four critical points:$(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}, 1/2, 1)$, $(-3\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}, -1/2, 1)$, $(3\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}, -1/2, 1)$, and $(-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}, 1/2, 1)$.

The bordered Hessian is

$$H^B = \begin{bmatrix} 0 & 0 & 0 & 2y & 2z \\ 0 & 0 & z & 0 & x \\ 0 & z & 0 & 0 & 1-\mu \\ 2y & 0 & 0 & -2\lambda & 1 \\ 2z & x & 1-\mu & 1 & -2\lambda \end{bmatrix}.$$

A critical point $(x^*, y^*, z^*, \lambda^*, \mu^*)$ is a local maximum $(-1)^k \det(H_k^B) > 0$ for all $k = m+1, \ldots, n$, where m is the number of constraints, n is the number of choice variables, and H_k^B is the matrix remaining after deleting all but the first $m + k$ rows and columns from H^B. In this problem, there are 3 choice variables and 2 constraints, so it must be that *k=3*. Since $m + k = 2 + 3 = 5$, it follows that $H_k^B = H^B$. Consider one critical point at a time.

1. $(x^*, y^*, z^*, \lambda^*, \mu^*) = (3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}, 1/2, 1)$. Evaluated at this critical point, the bordered Hessian is

$$H^B = \begin{bmatrix} 0 & 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 3\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 0 & 1-\mu \\ \sqrt{2} & 0 & 0 & -1 & 1 \\ \sqrt{2} & 3\sqrt{2} & 1-\mu & 1 & -1 \end{bmatrix}.$$

It follows that $(-1)^5 \det(H^B(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}, 1/2, 1)) = 4 > 0$. It follows that this critical point is a maximum.

2. $(x^*, y^*, z^*, \lambda^*, \mu^*) = (-3\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}, -1/2, 1)$. Using the same procedure, we see that this critical point is not a maximum.

3. $(x^*, y^*, z^*, \lambda^*, \mu^*) = (3\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}, -1/2, 1)$. Using the same procedure, we see that this critical point is also not a maximum.

4. $(x^*, y^*, z^*, \lambda^*, \mu^*) = (-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}, 1/2, 1)$. Using the same procedure, we see that this critical is also a maximum.

Problem 8.

$a)$

$$L = 50x^{1/2}y^2 - \lambda(x + y - 80).$$

$$L_x = 25x^{-1/2}y^2 - \lambda = 0,$$

$$L_y = 100x^{1/2}y - \lambda = 0.$$

Dividing,

$$4x/y = 1 \implies y = 4x$$

$$\implies x = 16, \quad y = 64, \quad Q = 819{,}200, \quad \lambda = 25{,}600.$$

$b)$ $Q^*(79) \approx Q^*(80) + \lambda^*(-1) = 819{,}200 - 25{,}600 = 793{,}600.$

Problem 9.

Our objective is to approximate the maximum and minimum distances from the origin to the ellipse given by the relation $x^2 + xy + 0.9y^2 = 3$. We can simplify the problem by looking for the maximum and minimum distances-squared; see the explanation in Problem 4. Based on Problem 4, we know that the maximum distance-squared from the origin to the ellipse given by the relation $x^2 + xy + y^2 = 3$ is 6 at the points $(x^*, y^*, \lambda^*) = (\sqrt{3}, -\sqrt{3}, 2)$ and $(x^*, y^*, \lambda^*) = (-\sqrt{3}, \sqrt{3}, 2)$. Furthermore, the minimum distance-squared from the origin to the ellipse given by the same relation is 2 at the points

$(x^*, y^*, \lambda^*) = (1, 1, 2/3)$ and $(x^*, y^*, \lambda^*) = (-1, -1, 2/3)$. Since the constraint in this problem is a minor variation of the constraint in Problem 4, we can use the Envelope Theorem. To do so, we first simplify the problem to maximize and minimize $x^2 + y^2$, rather than $\sqrt{x^2 + y^2}$. Next, we parameterize the problem by saying that we are looking for the maximum and minimum distances from the origin to the ellipse $x^2 + xy + by^2 = 3$. In either case, the Lagrangian is

$$L(x, y, \lambda, b) = x^2 + y^2 - \lambda(x^2 + xy + by^2 - 3).$$

Let f^* be either the maximum or minimum value of the objective function, depending on what we are looking for; in this case $f^* = f(x^*, y^*) = (x^*)^2 + (y^*)^2$. The Envelope Theorem says that

$$\frac{\partial f^*}{\partial b} = \frac{\partial L}{\partial b}(x^*, y^*, \lambda^*) = -(\lambda^*)(y^*)^2.$$

We can approximate the maximum and minimum distance to the ellipse we are interested in as $f^* + \frac{df^*}{db}$, where

$$\frac{df^*}{db} = \frac{\partial f^*}{\partial b} db.$$

We can see that $db = 0.9 - 1 = -0.1$. For the maximum,

$$\frac{\partial f^*}{\partial b} = \frac{\partial L}{\partial b}(-\sqrt{3}, \sqrt{3}, 2) = -(2)(\sqrt{3})^2 = -6,$$

which implies that

$$\frac{df^*}{db} \equiv \frac{\partial f^*}{\partial b} db = (-6)(-0.1) = 0.6.$$

Furthermore, this is the case regardless of which maximum we use. It follows that the maximum distance-squared is about 6+0.6=6.6. A similar calculation shows that the minimum distance-squared is about 2+0.2=2.2. Finally, we can take the square root of the distance-squared to see that the maximum distance is about 2.57 and the minimum distance is about 1.48.

Problem 10.

The expected value of loss due to a surge is

$$(0.01)(\$400) + (0.02)(\$200) + (0.1)(\$100) = \$18.$$

Thus, the most the entrepreneur would be willing to pay for a surge protector is \$18.

Problem 11.

(a) The probability a second procedure is required is 0.05 + 0.14 = 0.19.

(b) The probability that someone whose corrective lens factor is minus 8 or less does not require a second procedure is

$$\frac{0.15}{0.15 + 0.14} \equiv 0.51.$$

(c) The marginal distribution function for X is f(x), where f(0) = 0.66 + 0.15 = 0.81 and f(1) = 0.05 + 0.14 = 0.19. The marginal distribution function for Y is g(y), where g(0) = 0.15 + 0.14 = 0.29 and g(1) = 0.66 + 0.05 = 0.71.

(d) X and Y are independent if $Prob(x, y) = Prob(x)Prob(y)$. We can see, for example, that $Prob(0, 0) = 0.15 \neq Prob(0)Prob(0) = (0.81)(0.29) = 0.2349$. Thus, X and Y are not independent. In other words, the probabilities of requiring a second procedure and of having a corrective lens factor of minus 8 or less are not independent.