

Homework 3 Solutions

ECON 4323/5301 Fall 2007

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1. Find the maximizer of $f(x, y) = x^2 + y^2$ subject to the constraints that $2x + y \leq 2$, $x \geq 0$, and $y \geq 0$.

$$\begin{aligned} \max \quad & x^2 + y^2 \\ \text{s. t.} \quad & 2x + y \leq 2, \\ & -x \leq 0, \\ & -y \leq 0. \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x, y, \mu_1, \mu_2, \mu_3) = x^2 + y^2 - \mu_1(2x + y - 2) + \mu_2x + \mu_3y.$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2\mu_1 + \mu_2 = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \mu_1 + \mu_3 = 0, \tag{2}$$

$$2x + y \leq 2, \tag{3}$$

$$x \geq 0, \tag{4}$$

$$y \geq 0, \tag{5}$$

$$\mu_1 \geq 0, \tag{6}$$

$$\mu_2 \geq 0, \tag{7}$$

$$\mu_3 \geq 0, \tag{8}$$

$$\mu_1(2x + y - 2) = 0, \tag{9}$$

$$\mu_2x = 0, \tag{10}$$

$$\mu_3y = 0. \tag{11}$$

To find the critical points, I consider all the cases that satisfy equations 9, 10, and 11.

(a) $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0.$

Equations 1 and 2 imply that $x = 0$ and $y = 0$. Thus, one critical point is $(x, y, \mu_1, \mu_2, \mu_3) = (0, 0, 0, 0, 0).$

(b) $\mu_1 = 0, \mu_2 = 0, y = 0.$

Equations 1 and 2 imply that $x = 0$ and $\mu_3 = 0$, so we rediscover the critical point found in case 1.

(c) $\mu_1 = 0, x = 0, \mu_3 = 0.$

Equations 1 and 2 imply that $y = 0$ and $\mu_2 = 0$, so we again rediscover the critical point found in case 1.

(d) $2x + y = 2, \mu_2 = 0, \mu_3 = 0.$

$2x + y = 2 \Rightarrow y = 2 - 2x$. Substituting for y and μ_3 in equation 2 implies that $\mu_1 = 4 - 4x$. Substituting for μ_1 in equation 1 and solving for x implies that $x = 4/5$. It then follows that $y = 2/5$ and $\mu_1 = 4/5$. Thus, a second critical point is $(x, y, \mu_1, \mu_2, \mu_3) = (4/5, 2/5, 4/5, 0, 0).$

(e) $\mu_1 = 0, x = 0, y = 0.$

Equations 1 and 2 imply that $\mu_2 = 0$ and $\mu_3 = 0$, so we again rediscover the critical point found in case 1.

(f) $2x + y = 2, x = 0, \mu_3 = 0.$

If $x = 0$, then $y = 2$. Substituting for y and μ_3 in equation 2 implies that $\mu_1 = 4$. Substituting for x and μ_1 in equation 1 implies that $\mu_2 = 8$. Thus, a third critical point is $(x, y, \mu_1, \mu_2, \mu_3) = (0, 2, 4, 8, 0).$

(g) $2x + y = 2, \mu_2 = 0, y = 0.$

If $y = 0$, then $x = 1$. Substituting for x and μ_2 in equation 1 implies that $\mu_1 = 1$. Substituting for μ_1 and y in equation 2 implies that $\mu_3 = 1$. Thus, a fourth critical point is $(x, y, \mu_1, \mu_2, \mu_3) = (1, 0, 1, 0, 1).$

(h) $2x + y = 2, x = 0, y = 0.$

Clearly, no critical point satisfies all three assumptions.

To characterize each of the critical points, we use the second order conditions.

(a) $(x, y, \mu_1, \mu_2, \mu_3) = (0, 0, 0, 0, 0).$

The bordered Hessian matrix is

$$H_B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $\det(H_B) = 4$, it has the same sign as $(-1)^2 = 1 > 0$. However, we can see that the largest $n - m = 2 - 0 = 2$ largest leading principal minors

do not alternate in sign. Thus, H_B is not negative definite, so the critical point is not a local maximum.

(b) $(x, y, \mu_1, \mu_2, \mu_3) = (4/5, 2/5, 4/5, 0, 0)$.

The bordered Hessian matrix is

$$H_B = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Since $\det(H_B) = -10$, which is not the same sign as $(-1)^2 = 1 > 0$, H_B is not negative definite. Thus, the critical point is not a local maximum.

(c) $(x, y, \mu_1, \mu_2, \mu_3) = (0, 2, 4, 8, 0)$.

The bordered Hessian matrix is

$$\begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

$\det(H_B) = 1$, which is the same sign as $(-1)^n = (-1)^2 = 1 > 0$. We also need to check that the $n - m = 2 - 2 = 0$ largest leading principal minors alternate in sign – this is trivially satisfied. Thus, H_B is negative definite and the critical point is a local maximum.

(d) $(x, y, \mu_1, \mu_2, \mu_3) = (1, 0, 1, 0, 1)$.

The bordered Hessian matrix is

$$H_B = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix}.$$

$\det(H_B) = 4$, which is the same sign as $(-1)^n = (-1)^2 = 1 > 0$. We also need to check that the $n - m = 2 - 2 = 0$ largest leading principal minors alternate in sign – this is trivially satisfied. Thus, H_B is negative definite and the critical point is a local maximum.

Of the two critical points that are local maxima, we can easily check which generates the largest value of the objective function:

(a) $(x, y, \mu_1, \mu_2, \mu_3) = (0, 2, 4, 8, 0)$.

$$f(0, 2) = 4.$$

(b) $(x, y, \mu_1, \mu_2, \mu_3) = (1, 0, 1, 0, 1)$.

$$f(1, 0) = 1.$$

Thus, $(x, y, \mu_1, \mu_2, \mu_3) = (0, 2, 4, 8, 0)$ maximizes the objective function subject to the constraints.

2. Determine if the following functions are convex, concave, or neither:

(a)

$$f(x) = 3e^x + 5x^4 - \ln x$$

$f(x) = 3e^x + 5x^4 - \ln(x)$, so $f'(x) = 3e^x + 20x^3 - 1/x$ and $f''(x) = 3e^x + 60x^2 + 1/x^2$. Since $f''(x) > 0$ for all x , f is convex.

(b)

$$f(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$$

$f(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$. It follows that

$$\frac{\partial f}{\partial x} = -6x + 2y + 3, \quad \frac{\partial f}{\partial y} = 2x - 2y - 4.$$

Thus, the Hessian matrix of second and cross-partial derivatives is

$$H = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}.$$

The first order ($k = 1$) principal minors are -6 and -2. The second order ($k = 2$) principal minor is 8. Since the odd order principal minors are less than or equal to zero and the even order principal minors are greater than or equal to zero, the Hessian is negative semidefinite. Thus, f is concave.

(c)

$$f(x, y, z) = 3e^x + 5y^4 - \ln z$$

$f(x, y, z) = 3e^x + 5y^4 - \ln(x)$. It follows that

$$\frac{\partial f}{\partial x} = 3e^x, \quad \frac{\partial f}{\partial y} = 20y^3, \quad \frac{\partial f}{\partial z} = -1/z.$$

Thus, the Hessian matrix is

$$\begin{pmatrix} 3e^x & 0 & 0 \\ 0 & 60y^2 & 0 \\ 0 & 0 & 1/z^2 \end{pmatrix}.$$

It is easy to see that all the principal minors are greater than or equal to zero, so the Hessian matrix is positive semidefinite. Thus, f is convex.

3. Determine which of the following functions are homogeneous. For the homogeneous functions, identify the degree of homogeneity.

(a)

$$f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^3$$

$f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^3$. It follows that $f(tx, ty) = 3(tx)^5(ty) + 2(tx)^2(ty)^4 - 3(tx)^3(ty)^3 = t^6f(x, y)$. Thus, f is homogeneous of degree 6.

(b)

$$f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^4$$

Using the same technique as in part (a), we can see that the function in part (b) is not homogeneous of any degree.

(c)

$$f(x, y) = x^{1/2}y^{-1/2} + 3xy^{-1} + 7$$

Using the same technique as in part (a), we can see that the function in part (c) is homogeneous of degree zero.

4. Write down the degree-one homogenization of each of the following functions:

(a)

$$f(x) = e^x$$

$F(x, z) = zf(x/z) = ze^{x/z}$ is homogeneous of degree (HOD) 1.

(b)

$$f(x) = \ln x$$

$F(x, z) = zf(x/z) = z \ln(x/z)$ is HOD 1.

(c)

$$f(x) = 5$$

$F(x, z) = zf(x/z) = 5z$ is HOD 1.

(d)

$$f(x, y) = x^2 + y^3$$

$F(x, y, z) = zf(x/z, y/z) = x^2/z + y^3/z^2$ is HOD 1.

5. Find the following integrals:

(a)

$$\int (x^3 + 2x - 3) dx = \frac{1}{4}x^4 + x^2 - 3x + C.$$

(b)

$$\int (x - 1)^2 dx = \frac{1}{3}(x - 1)^3 + C.$$

(c)

$$\int (x - 1)(x + 2) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C.$$

(d)

$$\int (x + 2)^3 dx = \frac{1}{4}(x + 2)^4 + C.$$

(e)

$$\int (e^{3x} - e^{2x} + e^x) dx = \frac{1}{2}e^{3x} - \frac{1}{2}e^{2x} + e^x + C.$$

(f)

$$\int \frac{x^3 - 3x + 4}{x} dx = \frac{1}{3}x^3 - 3x + 4 \ln(|x|) + C.$$

6. Evaluate the following integrals:

(a)

$$\int_1^3 \frac{3x}{10} dx = \frac{3x^2}{20} \Big|_1^3 = \frac{3(3)^2}{20} - \frac{3(1)^2}{20} = \frac{24}{20} = \frac{6}{5}.$$

(b)

$$\int_{-3}^{-1} x^2 dx = 26/3.$$

(c)

$$\int_0^1 \alpha e^{\beta x} dx, \beta \neq 0 = \frac{\alpha(e^\beta - 1)}{\beta}.$$

(d)

$$\int_{-2}^{-1} \frac{1}{y} dy = -\ln(2).$$

7. Evaluate the following integrals:

(a)

$$\int_0^5 (x + x^2) dx = \frac{325}{6}.$$

(b) For $x < 0$, $e^x - e^{-x} < 0$, so the area under the curve is

$$-\int_{-2}^0 (e^x - e^{-x}) dx + \int_0^2 (e^x - e^{-x}) dx = 2(e^2 + e^{-2} - 2).$$

Otherwise,

$$\int_{-2}^2 (e^x - e^{-x}) dx = 0.$$

(c)

$$\int_2^{10} \frac{dx}{x-1} = \ln(|x-1|) \Big|_2^{10} = \ln(9) - \ln(1) = \ln(9).$$

(d) If $u = x^2$, then $du/dx = 2x$, which implies that $du = 2x dx$. It follows that

$$\int_0^1 2xe^{x^2} dx = \int_0^1 e^u du = e^u \Big|_0^1 = e^1 - e^0 = e - 1.$$

The limits of integration do not change because $x = 0$ implies that $u = 0^2 = 0$ and $x = 1$ implies that $u = 1^2 = 1$.

(e) For $x < 1$, $(x-1)^3 < 0$, so the area under the curve is

$$-\int_{-4}^1 (x-1)^3 dx + \int_1^4 (x-1)^3 dx = 353/2.$$

Otherwise,

$$-\int_{-4}^4 (x-1)^3 dx = -136.$$

(f)

$$\int_1^2 (x^5 + x^{-5}) dx = 687/64.$$

8. Find:

(a)

$$\frac{d}{dt} \int_0^t x^2 dx = t^2 - 0^2 = t^2.$$

(b)

$$\frac{d}{dt} \int_t^3 e^{-x^2} dx = - \int_3^t e^{-x^2} dx = -e^{-t^2}.$$

(c)

$$\begin{aligned} \frac{d}{dt} \int_{-t}^t \frac{1}{\sqrt{x^4+1}} dx &= - \int_0^{-t} \frac{1}{\sqrt{x^4+1}} dx + \int_0^t \frac{1}{\sqrt{x^4+1}} dx \\ &= - \frac{1}{\sqrt{(-t)^4+1}}(-1) + \frac{1}{\sqrt{t^4+1}} \\ &= \frac{2}{\sqrt{t^4+1}}. \end{aligned}$$

9. Evaluate the following:

(a) Let $u = \ln(x+2)$ and $dv = x dx$. Then $du = 1/(x+2)$ and $v = x^2/2$. It follows that

$$\int_{-1}^1 x \ln(x+2) dx = \frac{1}{2} x^2 \ln(x+2) \Big|_{-1}^1 - \frac{1}{2} \int_{-1}^1 \frac{x^2}{x+2} dx.$$

We can see that

$$\frac{1}{2} x^2 \ln(x+2) \Big|_{-1}^1 = \frac{1}{2} (\ln(3) - 0) = \frac{1}{2} \ln(3).$$

Furthermore,

$$\frac{x^2}{x+2} = \frac{x^2}{x+2} - \frac{4}{x+2} + \frac{4}{x+2} = \frac{x^2-4}{x+2} + \frac{4}{x+2} = \frac{(x+2)(x-2)}{x+2} + \frac{4}{x+2} = x-2 + \frac{4}{x+2}.$$

It follows that

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 \frac{x^2}{x+2} dx &= \frac{1}{2} \int_{-1}^1 x - 2 + \frac{4}{x+2} dx \\ &= \frac{x^2}{4} - x + 2 \ln(|x+2|) \Big|_{-1}^1 \\ &= \frac{1}{4} - 1 + 2 \ln(3) - \left(\frac{1}{4} + 1 + 2 \ln(1) \right) \\ &= 2 \ln(3) - 2. \end{aligned}$$

Thus,

$$\int_{-1}^1 x \ln(x+2) dx = \frac{1}{2} \ln(3) - (2 \ln(3) - 2) = 2 - (3/2) \ln(3).$$

(b)

$$\int_0^2 x2^x dx$$

Omitted.

(c)

$$\int_0^1 x^2 e^x dx$$

We did this one in class.

10. Find the following integrals by means of an appropriate substitution:

(a) Let $u = 2x^2 + 3$, so $du = 4xdx$. Then

$$\int x(2x^2 + 3)^5 dx = \frac{1}{4} \int u^5 du = \frac{1}{24} u^6 + C = \frac{1}{24} (2x^2 + 3)^6 + C.$$

(b) Let $u = x^3 + 2$, so $du = 3x^2 dx$. Then

$$\int x^2 e^{x^3+2} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3+2} + C.$$

(c) Let $u = x^2 - x + 8$, so $du = 2x - 1 dx$. Then

$$\int \frac{2x-1}{x^2-x+8} dx = \int du/u = \ln(|u|) + C = \ln(|x^2 - x + 8|) + C.$$

(d) Let $u = x + 1$, so $x = u - 1$ and $du = dx$. Then

$$\begin{aligned} \int x\sqrt{1+x} dx &= \int (u-1)u^{1/2} du \\ &= \int u^{3/2} - u^{1/2} du \\ &= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C \\ &= \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C. \end{aligned}$$

(e) Let $u = 1 + x^2$, so $du = 2xdx$ and $x^2 = u - 1$. Then

$$\begin{aligned} \int \frac{x^3}{(1+x^2)^3} dx &= \frac{1}{2} \int \frac{u-1}{u^3} du \\ &= \frac{1}{2} \left(-u^{-1} - \frac{u^{-2}}{-2} \right) + C \\ &= \frac{1}{2} \left(\frac{1}{2u^2} - \frac{1}{u} \right) + C \\ &= \frac{1}{2} \left(\frac{1}{2(1+x^2)^2} - \frac{1}{(1+x^2)} \right) + C. \end{aligned}$$

(f) Let $u = 4 - x^3$, so $du = -3x^2 dx$ and $x^3 = 4 - u$. Then

$$\begin{aligned} \int x^5 \sqrt{4 - x^3} dx &= -\frac{1}{3} \int (4 - u) u^{1/2} du \\ &= -\frac{1}{3} \int 4u^{1/2} - u^{3/2} du \\ &= -\frac{1}{3} \left(\frac{4u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2} \right) + C \\ &= \frac{2u^{5/2}}{15} - \frac{8u^{3/2}}{9} + C \\ &= \frac{2(4 - x^3)^{5/2}}{15} - \frac{8(4 - x^3)^{3/2}}{9} + C. \end{aligned}$$

11. Find an expression for $F'(x)$ when

(a) If $F(x) = \int_1^2 \frac{e^{xt}}{t} dt$, $x \neq 0$, then

$$F'(x) = \int_1^2 \frac{d}{dx} \left(\frac{e^{xt}}{t} \right) dt = \int_1^2 e^{xt} dt = \frac{1}{x} (e^{2x} - e^x).$$

(b) If $F(x) = \int_1^e \ln(xt) dt$, $x > 0$, then

$$F'(x) = \int_1^e \frac{d}{dx} (\ln(xt)) dt = \int_1^e \frac{1}{x} dt = \frac{1}{x} (e - 1).$$

(c) If $F(x) = \int_0^1 \frac{e^{-t}}{1+xt} dt$, $x > -1$, then

$$F'(x) = \int_0^1 \frac{d}{dx} \left(\frac{e^{-t}}{1+xt} \right) dt = \int_0^1 \frac{-te^t}{(1+xt)^2} dt.$$

(d) If $F(x) = \int_3^8 \frac{t^2}{(1-xt)^2} dt$, $x > 1/3$, then

$$F'(x) = \int_3^8 \frac{d}{dx} \left(\frac{t^2}{(1-xt)^2} \right) dt = \int_3^8 \frac{d}{dt} \frac{2t^3}{(1-xt)^3} dt.$$

12. Use Leibniz's Formula to find an expression for $F'(x)$ when

(a) If $F(x) = \int_0^{2x} t^3 dt$, then

$$F'(x) = \int_0^{2x} \frac{d}{dx} (t^3) dt + (2x)^3 (2) - (0)^3 (0) = \int_0^{2x} 0 dt + 16x^3 = 16x^3.$$

(b) If $F(x) = \int_0^x (x^2 + t^3)^2 dt$, then

$$\begin{aligned} F'(x) &= \int_0^x \frac{d}{dx} (x^2 + t^3)^2 dt + (x^2 + x^3)^2(1) - (x^2 + 0^3)^2(0) \\ &= \int_0^x 2(x^2 + t^3)(2x) dt + (x^2 + x^3)^2 \\ &= \int_0^x 4x^3 + 4xt^3 dt + (x^2 + x^3)^2 \\ &= 4x^4 + x^5 + (x^2 + x^3)^2 \\ &= 5x^4 + 3x^4 + x^6. \end{aligned}$$