

ECON 4323/5301 Homework 4 Solutions

Fall 2007

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1. Find the general solutions to the following linear differential equations:

(a)

$$\frac{dy}{dt} + y = 10,$$

(b)

$$\frac{dy}{dt} - 3y = 27,$$

(c)

$$4\frac{dy}{dt} + 5y = 100.$$

These are all first order linear differential equations with constant coefficient and constant term. The general form of such equations is

$$\frac{dy}{dt} + ay = b. \tag{1}$$

The general solution is

$$y(t) = y_c(t) + y_p(t),$$

where $y_c(t)$ is the complementary function and $y_p(t)$ is the particular integral. The complementary function is the general solution to the homogenous version of equation 1,

$$\frac{dy}{dt} + ay = 0,$$

so

$$y_c(t) = Ae^{-at},$$

where A is an undetermined coefficient. The particular integral is

$$y_p(t) = \frac{b}{a}.$$

Thus,

$$y(t) = Ae^{-at} + \frac{b}{a}.$$

For part (a), $a = 1$ and $b = 10$, so

$$y(t) = Ae^{-t} + 10.$$

Similarly, the solution to part (b) is $y(t) = Ae^{3t} - 9$, and the solution to part (c) is $y(t) = Ae^{-5t/4} + 20$.

2. Find the general solutions to the following differential equations:

(a)

$$t \frac{dy}{dt} + 2y + t = 0, t \neq 0,$$

(b)

$$\frac{dy}{dt} - \frac{1}{t}y = t, t > 0,$$

(c)

$$\frac{dy}{dt} - \frac{t}{t^2 - 1}y = t, t > 1,$$

(d)

$$\frac{dy}{dt} - \frac{2}{t}y + \frac{2a^2}{t} = 0, t > 0.$$

These are all first order linear differential equations with variable coefficient and variable term. The general form of such equations is

$$\frac{dy}{dt} + u(t)y = w(t). \quad (2)$$

The general solution is

$$y(t) = e^{-\int u(t)dt} \left(A + \int w(t)e^{\int u(t)dt} dt \right),$$

where A is an undetermined coefficient. For part (a) we first rearrange the equation,

$$\frac{dy}{dt} + \frac{2}{t}y = -1.$$

It follows that $u(t) = 2/t$ and $w(t) = -1$. Then

$$\int u(t)dt = \int 2/t dt = 2 \ln(|t|) = \ln(t^2).$$

It follows that

$$e^{-\int u(t)dt} = e^{-\ln(t^2)} = e^{\ln(t^{-2})} = t^{-2}$$

and

$$\int w(t)e^{\int u(t)dt} dt = \int (-1)e^{\ln(t^2)} dt = -\int t^2 dt = -t^3/3.$$

Thus,

$$y(t) = t^{-2} \left(A - t^3/3 \right) = At^{-2} - t/3.$$

Similar processes demonstrate that for part (b), $y(t) = At + t^2$, for part (c), $y(t) = A(t^2 - 1)^{1/2} + t^2 - 1$, and for part (d), $y(t) = At^2 + a^2$.

3. Solve the differential equation

$$1 + \left(2 + \frac{t}{y} \right) \frac{dy}{dt} = 0, t > 0, y > 0. \quad (3)$$

You probably start by trying to put this in the form of a linear first order differential equation with variable coefficient and variable term, but it never really works out. The next option is to see if this is an

exact differential equation. Recall that a differential equation is exact if it has the form

$$M(y, t) \frac{dy}{dt} + N(y, t) = 0,$$

where $\partial M/\partial t = \partial N/\partial y$. For equation 3, $M(y, t) = 2+t/y$, so $\partial M/\partial t = 1/y$, and $N(y, t) = 1$, so $\partial N/\partial y = 0$. Clearly, $\partial M/\partial t \neq \partial N/\partial y = 0$. Since equation 3 is not exact, we will look for an integrating factor. We can see that

$$\frac{\partial M/\partial t - \partial N/\partial y}{N} = \frac{1/y - 0}{1} = 1/y$$

is a function of y only, so

$$\beta(y) = e^{\int \frac{\partial M/\partial t - \partial N/\partial y}{N} dy} = e^{\int 1/y dy} = e^{\ln(|y|)} = y$$

is an integrating factor. Multiplying equation 3 by $\beta(y) = y$, we have

$$y + (2y + t) \frac{dy}{dt} = 0, t > 0, y > 0. \quad (4)$$

For equation 4, $M(y, t) = 2y + t$, with $\partial M/\partial t = 1$, and $N(y, t) = y$, with $\partial N/\partial y = 1$. Thus, equation 4 is exact, so we can solve for $y(t)$ using the technique for solving exact differential equations. The first step is to write

$$F(y, t) = \int 2y + t dy + \psi(t) = y^2 + ty + \psi(t).$$

The next step is to differentiate F with respect to t ,

$$\frac{\partial F}{\partial t} = y + \psi'(t),$$

and set $\partial F/\partial t$ equal to $N(y, t)$,

$$\frac{\partial F}{\partial t} = y + \psi'(t) = y = N(y, t).$$

It follows that $\psi'(t) = 0$. The third step is to integrate $\psi'(t)$ with respect to t to determine $\psi(t)$. Since

$$\int 0 dt = k,$$

where k is a constant, we have $\psi(t) = k$. The last step is to substitute for $\psi(t)$ in the equation for $F(y, t)$. We have

$$F(y, t) = y^2 + ty + k.$$

Setting $F(y, t) = C$ implicitly defines y as a function of t .

4. Solve the following Bernoulli equations assuming $t > 0$, $y > 0$:

(a)

$$t \frac{dy}{dt} + 2y = ty^2,$$

(b)

$$\frac{dy}{dt} = 4y + 2e^t \sqrt{y},$$

(c)

$$t \frac{dy}{dt} + y = y^2 \ln t.$$

These can all be turned into Bernoulli equations, which have the form

$$\frac{dy}{dt} + R(t)y = T(t)y^m.$$

Rearranging the equation in part (a) yields

$$\frac{dy}{dt} + 2y/t = y^2.$$

In this case, $R(t) = 2/t$, $T(t) = 1$, and $m = 2$. Let $z = y^{1-m} = y^{1-2} = y^{-1}$. Then

$$\frac{dz}{dt} - 2z/t = -1. \tag{5}$$

Equation 5 is a first order linear differential equation with variable coefficient and variable term, and so we can solve for $z(t)$ using the method described in problem 2. Doing so generates $z(t) = At^2 + t$. It follows that $y(t) = 1/(At^2 + t)$.

Similar arguments imply that the solution to part (b) is $y(t) = (Ae^{2t} - e^t)^2$ and the solution to part (c) is $y(t) = 1/(1 + \ln(t) + At)$.

5. An economic growth model by Haavelmo (1954) leads to the differential equation

$$\frac{dK}{dt} = \gamma_1 b K^\alpha + \gamma_2 K,$$

where γ_1 , γ_2 , b , and α are positive constants, $\alpha \neq 1$, and $K = K(t)$ is the unknown function. The equation is separable, but solve it as a Bernoulli equation.

Using the method described in the previous problem, we find that

$$K(t) = [Ae^{\gamma_2(1-\alpha)t} - \gamma_1 b / \gamma_2]^{1/(1-\alpha)}.$$

6. A study of the optimal exhaustion of a natural resource uses the equation

$$\frac{d^2 y}{dt^2} - \frac{2-\alpha}{1-\alpha} a \frac{dy}{dt} + \frac{a^2}{1-\alpha} y = 0, \quad (6)$$

where $\alpha \neq 0$, $\alpha \neq 1$, and $a \neq 0$. Prove that $u_1 = e^{at}$ and $u_2 = e^{at/(1-\alpha)}$ are both solutions. What is the general solution?

To show that $u_1 = e^{at}$ is a solution to 6, we first calculate

$$\frac{du_1}{dt} = ae^{at} \text{ and } \frac{d^2 u_1}{dt^2} = a^2 e^{at}.$$

Substituting into 6, we have

$$\begin{aligned} a^2 e^{at} - \frac{2-\alpha}{1-\alpha} a (ae^{at}) + \frac{a^2}{1-\alpha} e^{at} &= e^{at} \left(a^2 - a^2 \frac{2-\alpha}{1-\alpha} + \frac{a^2}{1-\alpha} \right) \\ &= e^{at} \left(a^2 \frac{1-\alpha}{1-\alpha} - a^2 \frac{2-\alpha}{1-\alpha} + \frac{a^2}{1-\alpha} \right) \\ &= \frac{e^{at}}{1-\alpha} (a^2(1-\alpha) - a^2(2-\alpha) + a^2) \\ &= \frac{e^{at}}{1-\alpha} (a^2 - \alpha a^2 - 2a^2 + \alpha a^2 + a^2) \\ &= \frac{e^{at}}{1-\alpha} (0) \\ &= 0. \end{aligned}$$

Thus, $u_1 = e^{at}$ is a solution to 6. A similar argument demonstrates that $u_2 = e^{at/(1-\alpha)}$ is also a solution to 6. It follows that the general solution to 6 is a linear combination of u_1 and u_2 , so $y(t) = Ae^{at} + Be^{at/(1-\alpha)}$.

7. Find the general solutions of the following equations:

(a)

$$\frac{d^2y}{dt^2} - 3y = 0,$$

(b)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 0,$$

(c)

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 8,$$

(d)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{5t},$$

(e)

$$\frac{d^2y}{dt^2} - y = e^{-t},$$

(f)

$$3\frac{d^2y}{dt^2} - 30\frac{dy}{dt} + 75y = 2t + 1.$$

The equations in parts (a), (b), and (c) are second order linear differential equations with constant coefficients and constant terms. The general form for these equations is

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2y = b.$$

The general solution is

$$y(t) = y_c(t) + y_p(t),$$

where $y_c(t)$ is the complementary function and $y_p(t)$ is the particular integral. For part (a), $a_1 = 0$, $a_2 = -3$ and $b = 0$. Thus, the particular integral is

$$y_p(t) = b/a_2 = -0/3 = 0.$$

To find the complementary function, we need to find

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

Substituting for a_1 and a_2 , we see that the roots are $(0 \pm \sqrt{0 - (4)(-3)})/2 = \pm\sqrt{12}/2 = \pm\sqrt{12}/4 = \pm\sqrt{3}$, so we can set $r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$. Since $r_1 \neq r_2$, the complementary function is

$$y_c(t) = Ae^{\sqrt{3}t} + Be^{-\sqrt{3}t}.$$

It follows that the solution to part (a) is $y(t) = y_c(t) + y_p(t) = Ae^{\sqrt{3}t} + Be^{-\sqrt{3}t}$.

Similar calculations show that the solution to part (b) is $y(t) = e^{-2t}(A \cos(2t) + B \sin(2t))$ and the solution to part (c) is $y(t) = Ae^{-3t} + Be^{2t} - 4/3$.

The equations for parts (d), (e), and (f) are second order linear differential equations with constant coefficients and variable terms. The general form for these equations is

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = f(t).$$

The general solution is

$$y(t) = y_c(t) + y_p(t),$$

where $y_c(t)$ is the complementary function and $y_p(t)$ is the particular integral. The complementary function for each equation is found using the same method used to find the complementary function for second order linear differential equations with constant coefficients and constant term. It follows that the complementary function for part (d) is

$$y_c(t) = Ae^{-t} + Be^{-2t}.$$

To find the particular integral for part (d), we notice that $f(t) = e^{5t}$. This has the form $f(t) = pe^{qt}$, where $p = 1$ and $q = 5$. We need to check that $q = 5$ is not a solution to $q^2 + 3q + 2 = 0$. In fact, the values of q that solve $q^2 + 3q + 2 = 0$ are $q = -1$ and $q = -2$. (This follows from the fact that $q^2 + 3q + 2 = (q + 1)(q + 2)$, so $q^2 + 3q + 2 = 0$ implies that $(q + 1)(q + 2) = 0$. The solutions follow.) As a result,

$$y_p(t) = \frac{p}{q^2 + a_1q + a_2}e^{qt} = \frac{1}{5^2 + (3)(5) + 2}e^{5t} = \frac{1}{42}e^{5t}.$$

Thus, the solution to part (d) is

$$y(t) = Ae^{-t} + Be^{-2t} + \frac{1}{42}e^{5t}.$$

Similar reasoning indicates that the solution to part (e) is $y(t) = Ae^t + Be^{-t} - (t/2)e^{-t}$ and the solution to part (f) is $y(t) = Ae^{5t} + Bte^{5t} + (2t/75) + 3/125$.