SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 04

DUE ON: 07/14/2008

QUESTIONS:

→ Find the inverse Laplace transform:

(1) $\mathcal{L}^{-1}\left[\dfrac{2s+1}{s^2+16}\right]$

(2) $\mathcal{L}^{-1}\left[\dfrac{2s+3}{(s-2)(s^2+1)}\right]$

(3) $\mathcal{L}^{-1}\left[\dfrac{3}{s^2+3s-10}\right]$

(4) $\mathcal{L}^{-1}\left[\dfrac{2s+1}{s^2+2s+4}\right]$

(5) $\mathcal{L}^{-1}\left[\dfrac{2s+3}{(s+5)^2+49}\right]$

(6) $\mathcal{L}^{-1}\left[\dfrac{s+4}{(s-1)(s+2)(s-3)}\right]$

→ Use Laplace transform to solve the given IVP.

(7) ~~(6)~~ $y'' + 2y' + y = 2e^{-t}$; $y(0) = 2$; $y'(0) = 1$

(8) ~~(7)~~ $y'' - y' - 2y = 10\cos t$; $y(0) = 0$; $y'(0) = -1$

(9) ~~(8)~~ $y'' - y = 8e^{t}\sin 2t$; $y(0) = 2$; $y'(0) = -2$

SOLUTIONS:

Find the inverse laplace transform:

(1) $L^{-1}\left[\dfrac{2s+1}{s^2+16}\right]$

Sol:- $L^{-1}\left[\dfrac{2s+1}{s^2+16}\right] = 2\,L^{-1}\left[\dfrac{s}{s^2+16}\right] + L^{-1}\left[\dfrac{1}{s^2+16}\right]$

$= 2\cos 4t + \frac{1}{4} L^{-1}\left[\dfrac{4}{s^2+16}\right]$

$\Rightarrow L^{-1}\left[\dfrac{2s+1}{s^2+16}\right] = 2\cos 4t + \frac{1}{4}\sin 4t$

(2) $L^{-1}\left[\dfrac{2s+3}{(s-2)(s^2+1)}\right]$

Sol:- $$\frac{2s+3}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

with $s=2$: $A = 7/5$

$s=0$: $C = -4/5$

$s^2=-1$: $B = -7/5$

$$\therefore \quad \frac{2s+3}{(s-2)(s^2+1)} = \frac{7/5}{s-2} + \frac{\left(-\frac{7}{5}s\right) - \frac{4}{5}}{s^2+1}$$

$$\therefore L^{-1}\left[\frac{2s+3}{(s-2)(s^2+1)}\right] = \frac{7}{5} L^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{5} L^{-1}\left[\frac{7s+4}{s^2+1}\right]$$

$$= \frac{1}{5}\left[7L^{-1}\left[\frac{1}{s-2}\right] - 7L^{-1}\left[\frac{s}{s^2+1}\right] - 4L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$\Rightarrow L^{-1}\left[\frac{2s+3}{(s-2)(s^2+1)}\right] = \frac{1}{5}\left\{7e^{2t} - 7\cos t - 4\sin t\right\}$$

Alternative method to find Partial fractions:

$$\frac{2s+3}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+c}{s^2+1}$$

$$\Rightarrow 2s+3 = A(s^2+1) + (Bs+c)(s-2)$$

$$= As^2 + A + Bs^2 - 2Bs + cs - 2c$$

$$\Rightarrow 2s+3 = (A+B)s^2 + (c-2B)s + (A-2c)$$

Comparing the like terms,

$$A+B=0 \; ; \quad c-2B=2 \; ; \quad A-2c=3$$

on solving,

$$A = \frac{7}{5} \; ; \; B = -7/5 \; ; \; c = \frac{-4}{5}.$$

(3) $L^{-1}\left[\dfrac{3}{s^2+3s-10}\right]$

Sol:- $\dfrac{3}{s^2+3s-10} = \dfrac{3}{(s+5)(s-2)}$

$$\therefore \frac{3}{(s+5)(s-2)} = \frac{A}{s+5} + \frac{B}{s-2}$$

with $s = -5$: $A = -3/7$

& with $s = 2$: $B = 3/7$

$$\therefore \frac{3}{(s+5)(s-2)} = \frac{-3/7}{s+5} + \frac{3/7}{s-2}$$

Now, $$L^{-1}\left[\frac{3}{(s+5)(s-2)}\right] = L^{-1}\left[\frac{-3/7}{s+5}\right] + L^{-1}\left[\frac{3/7}{s-2}\right]$$

$$= -\frac{3}{7} L^{-1}\left[\frac{1}{s+5}\right] + \frac{3}{7} L^{-1}\left[\frac{1}{s-2}\right]$$

$$\therefore L^{-1}\left[\frac{3}{(s+5)(s-2)}\right] = -\frac{3}{7} e^{-5t} + \frac{3}{7} e^{2t}$$

(4) $L^{-1}\left[\dfrac{2s+1}{s^2+2s+4}\right]$

Sol:- $$\frac{2s+1}{s^2+2s+4} = \frac{2s+1}{(s+1)^2+3}$$

$$= \frac{2(s+1)-1}{(s+1)^2+3}$$

Now, $$L^{-1}\left[\frac{2s+1}{s^2+2s+4}\right] = L^{-1}\left[\frac{2(s+1)-1}{(s+1)^2+3}\right]$$

$$= 2L^{-1}\left[\frac{(s+1)}{(s+1)^2+3}\right] - L^{-1}\left[\frac{1}{(s+1)^2+3}\right]$$

$$= 2e^{-t}\cos\sqrt{3}t - \frac{1}{\sqrt{3}}L^{-1}\left[\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right]$$

$$= 2e^{-t}\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\left[e^{-t}\sin\sqrt{3}t\right]$$

$$\therefore L^{-1}\left[\frac{2s+1}{(s+1)^2+3}\right] = 2e^{-t}\cos\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t.$$

(5) $$L^{-1}\left[\frac{2s+3}{(s+5)^2+49}\right]$$

Sol:- $$\frac{2s+3}{(s+5)^2+49} = \frac{2(s+5)-10+3}{(s+5)^2+49}$$

$$= \frac{(2s+10)-7}{(s+5)^2+49}$$

Now, $$L^{-1}\left[\frac{2s+3}{(s+5)^2+49}\right] = L^{-1}\left[\frac{2(s+5)-7}{(s+5)^2+49}\right]$$

$$= 2L^{-1}\left[\frac{s+5}{(s+5)^2+49}\right] - L^{-1}\left[\frac{7}{(s+5)^2+49}\right]$$

$$\therefore L^{-1}\left[\frac{2s+3}{(s+5)^2+49}\right] = 2e^{-5t}\cos 7t - e^{-5t}\sin 7t$$

$$= e^{-5t}\left[2\cos 7t - \sin 7t\right]$$

(6) $L^{-1}\left[\dfrac{s+4}{(s-1)(s+2)(s-3)}\right]$

Sol:- $\dfrac{s+4}{(s-1)(s+2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s+2} + \dfrac{C}{s-3}$

Now, with $s=1$: $A = -5/6$

with $s=-2$: $B = 2/15$

with $s=3$: $C = 7/10$

$\therefore \dfrac{s+4}{(s-1)(s+2)(s-3)} = \dfrac{(-5/6)}{s-1} + \dfrac{(2/15)}{s+2} + \dfrac{(7/10)}{s-3}$

$\therefore L^{-1}\left[\dfrac{s+4}{(s-1)(s+2)(s-3)}\right] = -\dfrac{5}{6} L^{-1}\left[\dfrac{1}{s-1}\right] + \dfrac{2}{15} L^{-1}\left[\dfrac{1}{s+2}\right] + \dfrac{7}{10} L^{-1}\left[\dfrac{1}{s-3}\right]$

$= -\dfrac{5}{6} e^{t} + \dfrac{2}{15} e^{-2t} + \dfrac{7}{10} e^{3t}$

$\therefore L^{-1}\left[\dfrac{s+4}{(s-1)(s+2)(s-3)}\right] = \dfrac{1}{30}\left[-25e^{t} + 4e^{-2t} + 21e^{3t}\right]$

→ Use Laplace transform to solve the given IVP:

(7) $y'' + 2y' + y = 2e^{-t}$; $y(0) = 2$; $y'(0) = 1$

Sol:- $L\{y'' + 2y' + y\} = L\{2e^{-t}\}$

$$\therefore \; L[y''] + 2L[y'] + L[y] = 2L[e^{-t}]$$

$$\Rightarrow \left[s^2 L[Y] - s\,y(0) - y'(0)\right] + 2\left[s\,L[Y] - y(0)\right] + L[y] = \frac{2}{s+1}$$

$$\Rightarrow s^2 Y[s] - 2s - 1 + 2s\,Y(s) - 4 + Y(s) = \frac{2}{s+1}$$

$$\Rightarrow Y[s]\left\{s^2 + 2s + 1\right\} - 2s - 5 = \frac{2}{s+1}$$

$$\Rightarrow Y[s]\left\{(s+1)^2\right\} = \frac{2}{s+1} + 2s + 5$$

$$\Rightarrow Y[s] = \frac{2}{(s+1)^3} + \frac{2s+5}{(s+1)^2} = \frac{2 + (2s+5)(s+1)}{(s+1)^3}$$

$$\Rightarrow Y[s] = \frac{2s^2 + 7s + 7}{(s+1)^3}$$

Now, $\dfrac{2s^2 + 7s + 7}{(s+1)^3} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{(s+1)^3}$

$$\boxed{\cancel{B \Rightarrow A(s+1)^2 + B}}$$

$$\Rightarrow A(s+1)^2 + B(s+1) + C = 2s^2 + 7s + 7$$

with $s = -1$; $C = 2$

$$\therefore \; A(s^2 + 2s + 1) + Bs + B + 2 = 2s^2 + 7s + 7$$

$$\Rightarrow As^2 + (2A + B)s + (A + B + 2) = 2s^2 + 7s + 7$$

Comparing the like terms,

$$A = 2 \; ; \; 2A + B = 7 \; ; \; A + B + 2 = 7$$

$$\therefore \; B = 3 \; ; \; C = 2$$

$$\therefore \; Y[s] = \frac{2s^2 + 7s + 7}{(s+1)^3} = \frac{2}{s+1} + \frac{3}{(s+1)^2} + \frac{2}{(s+1)^3} \quad - [A]$$

Now, applying inverse Laplace transform on both sides,

$$L^{-1}[Y[s]] = 2L^{-1}\left[\frac{1}{s+1}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{2}{(s+1)^3}\right]$$

$$\Rightarrow y(t) = 2e^{-t} + 3te^{-t} + t^2 e^{-t}$$

Altenative method to solve $Y[s]$:

$$Y[s] = \frac{2}{(s+1)^3} + \frac{2s+5}{(s+1)^2}$$

$$= \frac{2}{(s+1)^3} + \frac{2(s+1) - 2 + 5}{(s+1)^2}$$

$$= \frac{2}{(s+1)^3} + \frac{2(s+1) + 3}{(s+1)^2}$$

$$\Rightarrow Y[s] = \frac{2}{(s+1)^3} + \frac{2}{(s+1)} + \frac{3}{(s+1)^2}$$

which is same as obtained in eq [A]

(8) $y'' - y' - 2y = 10\cos t \; ; \; y(0) = 0 \; ; \; y'(0) = -1$

Sol: $L[y'' - y' - 2y] = 10\,L[\cos t]$

$$\Rightarrow \left[s^2 L[y] - s\,y(0) - y'(0)\right] - \left[s\,L[y] - y(0)\right] - 2L[y] = \frac{(10)s}{s^2+1}$$

$$\Rightarrow Y[s]\{s^2 - s - 2\} + 1 = \frac{10s}{s^2+1}$$

$$\Rightarrow Y[s] = \frac{10s}{(s^2+1)(s^2-s-2)} - \frac{1}{(s^2-s-2)}$$

$$= \frac{10s}{(s^2+1)(s+1)(s-2)} - \frac{1}{(s+1)(s-2)}$$

$$\Rightarrow Y[s] = \frac{10s - (s^2+1)}{(s^2+1)(s+1)(s-2)}$$

Now, $$\frac{10s - (s^2+1)}{(s^2+1)(s+1)(s-2)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-2}$$

with $s = -1$: $C = 2$

$s = 2$: $D = 1$

$s = 0$: $B = -1$

$s = 1$: $A = -3$

$$\therefore\ Y[s] = \frac{-3s-1}{s^2+1} + \frac{2}{s+1} + \frac{1}{s-2}$$

applying inverse laplace transforms on both sides,

$$L^{-1}\{Y[s]\} = -3L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left\{\frac{1}{s^2+1}\right\} + 2L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left\{\frac{1}{s-2}\right]$$

$$\Rightarrow\ y(t) = -3\cos t - \sin t + 2e^{-t} + e^{2t}$$

(9) $y'' - y = 8e^{t}\sin 2t$; $y(0) = 2$; $y'(0) = -2$

Sol:- $L\{y'' - y\} = 8L[e^{t}\sin 2t]$

$$\Rightarrow \left[s^2Y[s] - sy(0) - y'(0)\right] - Y[s] = \frac{16}{(s-1)^2+4}$$

$$\Rightarrow\ Y[s]\{s^2-1\} - 2s + 2 = \frac{16}{(s-1)^2+4}$$

$$\Rightarrow\ Y[s]\{s^2-1\} = \frac{16}{s^2-2s+1+4} + 2s - 2$$

$$\Rightarrow\ Y[s] = \frac{16}{(s^2-2s+5)(s^2-1)} + \frac{2s-2}{s^2-1}$$

$$= \frac{16}{(s^2-2s+5)(s+1)(s-1)} + \frac{2\cancel{(s-1)}}{(s+1)\cancel{(s-1)}}$$

$$\Rightarrow\ Y[s] = \frac{16}{(s+1)(s-1)(s^2-2s+5)} + \frac{2}{s+1}$$

Now, ~~for~~

$$\frac{16}{(s+1)(s-1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2-2s+5}$$

with $s=1$: $B=2$

$s=-1$: $A=-1$

$s=0$: $D=-1$

$s=2$: $C=-1$

$$\therefore \frac{16}{(s+1)(s-1)(s^2-2s+5)} = \frac{-1}{s+1} + \frac{2}{s-1} - \frac{(s+1)}{s^2-2s+5}$$

$$\therefore Y[s] = \frac{-1}{s+1} + \frac{2}{s-1} - \frac{(s+1)}{(s-1)^2+4} + \frac{2}{s+1}$$

$$= \frac{2}{s-1} + \frac{1}{s+1} - \frac{(s-1+2)}{(s-1)^2+4}$$

$$\Rightarrow Y[s] = \frac{2}{s-1} + \frac{1}{s+1} - \frac{s-1}{(s-1)^2+4} - \frac{2}{(s-1)^2+4}$$

Now applying inverse laplace transforms on both sides,

$$y(t) = 2L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s-1}{(s-1)^2+4}\right] - L^{-1}\left[\frac{2}{(s-1)^2+4}\right]$$

$$\Rightarrow y(t) = 2e^{t} + e^{-t} - e^{t}\left[\cos 2t + e^{t}\sin 2t\right]$$