SUB: MAE 3360

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~~SOLUTIONS TO ASSIGNMENT #06~~

~~DUE ON: 03/28/2007~~

QUESTIONS:

ADDITIONAL PROBLEMS ON NUMERICAL METHODS [PART - B]

EXERCISE : 6.1

Given the initial value problems, use the improved Euler's method to obtain a four-decimal approximation to the indicated value. First use $h=0.1$ & then $h=0.05$

(2) $y' = 4x - 2y$; $y(0) = 2$; $y(0.5)$

(4) $y' = x^2 + y^2$; $y(0) = 1$; $y(0.5)$

EXERCISE : 6.2

Use the RK4 method with $h=0.1$ to obtain a four-decimal approximation to the indicated value.

(4) $y' = 4x - 2y$; $y(0) = 2$; $y(0.5)$

EXERCISE : 6.4

(2) Use Euler's method to approximate $y(1.2)$, where $y(x)$ is the solution of initial value problem

$$x^2 y'' - 2xy' + 2y = 0 \; ; \; y(1) = 4 \; ; \; y'(1) = 9 \; , \; x > 0$$

Use $h=0.1$. Find the analytic sol. of the problem and compare the actual value of $y(1.2)$ with y_2.

(4) Repeat the above problem [i.e Prob. no. (2) in 6.4] using the RK4 method. First use $h=0.2$ & then use $h=0.1$

SOLUTIONS

EXERCISE: 6.1

(2) $y' = 4x - 2y$; $y(0) = 2$; $y(0.5)$ $h = 0.1$ & $h = 0.05$

Sol:- Given that $y(0) = 2$

$$\Rightarrow x_0 = 0 \; ; \; y_0 = 2$$

$$f(x_n, y_n) = 4x_n - 2y_n$$

For $h = 0.1$:

$n = 0$: $y_1^* = y_0 + h f(x_0, y_0) = 2 + (0.1)[4(0) - 2(2)] = 1.6$

$$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}\right] \; ; \; x_1 = x_0 + h = 0.1$$

$$= 2 + (0.1)\left[\frac{4(0) - 2(2) + 4(0.1) - 2(1.6)}{2}\right]$$

$$\Rightarrow y_1 = 1.6600$$

$n = 1$: $y_2^* = 1.6600 + (0.1)[4(0.1) - 2(1.6600)] = 1.3680$

$x_2 = 0.2$; $y_2 = 1.6600 + (0.1)\left[\frac{4(0.1) - 2(1.6600) + [4(0.2) - 2(1.3680)]}{2}\right]$

$$\Rightarrow y_2 = 1.4172$$

$n = 2$: $y_3^* = 1.4172 + (0.1)[4(0.2) - 2(1.4172)] = 1.2138$

$x_3 = 0.3$; $y_3 = 1.4172 + (0.1)\left[\frac{4(0.2) - 2(1.4172) + 4(0.3) - 2(1.2138)}{2}\right]$

$$\Rightarrow y_3 = 1.2541$$

$n = 3$: $y_4^* = 1.2541 + (0.1)[4(0.3) - 2(1.2541)] = 1.1233$

$x_4 = 0.4$; $y_4 = 1.2541 + (0.1)\left[\frac{4(0.3) - 2(1.2541) + 4(0.4) - 2(1.1233)}{2}\right]$

$\Rightarrow y_4 = 1.1564$

$\underline{n=4}$: $y_5^* = 1.1564 + (0.1)\left[4(0.4) - 2(1.1564)\right] = 1.08512$

$x_5 = 0.5$; $y_5 = 1.1564 + (0.1)\left[\frac{4(0.4) - 2(1.1564) + 4(0.5) - 2(1.08512)}{2}\right]$

$\Rightarrow y_5 = 1.1122$

For $\underline{h = 0.05}$

$x_0 = 0$; $y_0 = 2$

$\underline{n=0}$: $y_1^* = y_0 + h f(x_0, y_0) = 2 + (0.05)\left[4(0) - 2(2)\right] = 1.8$

$x_1 = x_0 + h = 0 + 0.05 = 0.05$

$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}\right]$

$= 2 + (0.05)\left[\frac{4(0) - 2(2) + 4(0.05) - 2(1.8)}{2}\right]$

$\Rightarrow y_1 = 1.8150$

$\underline{n=1}$: $y_2^* = 1.8150 + (0.05)\left[4(0.05) - 2(1.8150)\right] = 1.6435$

$x_2 = 0.1$; $y_2 = 1.8150 + (0.05)\left[\frac{4(0.05) - 2(1.8150) + 4(0.1) - 2(1.6435)}{2}\right]$

$\Rightarrow y_2 = 1.6571$

$\underline{n=2}$: $y_3^* = 1.6571 + (0.05)\left[4(0.10) - 2(1.6571)\right] = 1.5114$

$x_3 = 0.2$; $y_3 = 1.6571 + (0.05)\left[\frac{4(0.10) - 2(1.6571) + 4(0.15) - 2(1.5114)}{2}\right]$

$\Rightarrow y_3 = 1.5237$

$\underline{n=3}$: $y_4^* = 1.5237 + (0.05)\left[4(0.15) - 2(1.5237)\right] = 1.4013$

$x_4 = 0.2$; $y_4 = 1.5237 + (0.05)\left[\dfrac{4(0.15) - 2(1.5237) + 4(0.2) - 2(1.4013)}{2}\right]$

$\Rightarrow y_4 = 1.4124$

$\underline{n=4}$: $y_5^* = 1.4124 + (0.05)\left[4(0.2) - 2(1.4124)\right] = 1.3112$

$x_5 = 0.25$; $y_5 = 1.4124 + (0.05)\left[\dfrac{4(0.2) - 2(1.4124) + 4(0.25) - 2(1.3112)}{2}\right]$

$\Rightarrow y_5 = 1.3212$

$\underline{n=5}$: $y_6^* = 1.3212 + (0.05)\left[4(0.25) - 2(1.3212)\right] = 1.2391$

$x_6 = 0.3$; $y_6 = 1.3212 + (0.05)\left[\dfrac{4(0.25) - 2(1.3212) + 4(0.3) - 2(1.2391)}{2}\right]$

$\Rightarrow y_6 = 1.2482$

$\underline{n=6}$: $y_7^* = 1.2482 + (0.05)\left[4(0.3) - 2(1.2482)\right] = 1.1834$

$x_7 = 0.35$; $y_7 = 1.2482 + (0.05)\left[\dfrac{4(0.3) - 2(1.2482) + 4(0.35) - 2(1.1834)}{2}\right]$

$\Rightarrow y_7 = 1.1916$

$\underline{n=7}$: $y_8^* = 1.1916 + (0.05)\left[4(0.35) - 2(1.1916)\right] = 1.1424$

$x_8 = 0.4$; $y_8 = 1.1916 + (0.05)\left[\dfrac{4(0.35) - 2(1.1916) + 4(0.4) - 2(1.1424)}{2}\right]$

$\Rightarrow y_8 = 1.1499$

$\underline{n=8}$: $y_9^* = 1.1499 + (0.05)\left[4(0.4) - 2(1.1499)\right] = 1.1149$

$x_9 = 0.45$; $y_9 = 1.1499 + (0.05)\left[\dfrac{4(0.4) - 2(1.1499) + 4(0.45) - 2(1.1149)}{2}\right]$

$\Rightarrow y_9 = 1.1217$

$\underline{n=9}$: $y_{10}^{*} = 1.1217 + (0.05)\left[4(0.45) - 2(1.1217)\right] = 1.0995$

$x_{10} = 0.5$; $y_{10} = 1.1217 + (0.05)\left[\frac{4(0.45) - 2(1.1217) + 4(0.5) - 2(1.0995)}{2}\right]$

$\Rightarrow y_{10} = 1.1056$

Ex: 6.1

(4) $y' = x^2 + y^2$; $y(0) = 1$; $y(0.5)$ $\quad h = 0.1$ & $h = 0.05$

Sol:- Given that $y(0) = 1 \Rightarrow x_0 = 0$ & $y_0 = 1$

$f(x_n, y_n) = x_n^2 + y_n^2$

For $h = 0.1$

$\underline{n=0}$: $y_1^{*} = y_0 + h f(x_0, y_0) = 1 + (0.1)\left[0^2 + 1^2\right] = 1.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^{*})}{2}\right]$

$\Rightarrow y_1 = 1 + (0.1)\left[\frac{0^2 + 1^2 + (0.1)^2 + (1.1)^2}{2}\right]$

$\Rightarrow y_1 = 1.1110$

$\underline{n=1}$: $y_2^{*} = 1.111 + (0.1)\left[(0.1)^2 + (1.111)^2\right] = 1.2354$

$x_2 = 0.2$; $y_2 = 1.111 + (0.1)\left[\frac{(0.1)^2 + (1.111)^2 + (0.2)^2 + (1.2354)^2}{2}\right]$

$\Rightarrow y_2 = 1.2515$

$\underline{n=2}$: $y_3^{*} = 1.2515 + (0.1)\left\{(0.2)^2 + (1.2515)^2\right] = 1.4122$

$$x_3 = 0.3; \quad y_3 = 1.2515 + (0.1)\left[\frac{(0.2)^2 + (1.2515)^2 + (0.3)^2 + (1.4122)^2}{2}\right]$$

$$\Rightarrow y_3 = 1.4361$$

$\underline{n=3}$: $\quad y_4^* = 1.4361 + (0.1)\left[(0.3)^2 + (1.4361)^2\right] = 1.6513$

$$x_4 = 0.4; \quad y_4 = 1.4361 + (0.1)\left[\frac{(0.3)^2 + (1.4361)^2 + (0.4)^2 + (1.6513)^2}{2}\right]$$

$$\Rightarrow y_4 = 1.6880$$

$\underline{n=4}$: $\quad y_5^* = 1.6880 + (0.1)\left[(0.4)^2 + (1.6880)^2\right] = 1.9889$

$$x_5 = 0.5; \quad y_5 = 1.6880 + (0.1)\left[\frac{(0.4)^2 + (1.6880)^2 + (0.5)^2 + (1.9889)^2}{2}\right]$$

$$\Rightarrow y_5 = 2.0488$$

$\underline{\text{For } h = 0.05}$; $y_0 = 1$ & $x_0 = 0$

$\underline{n=0}$: $\quad y_1^* = y_0 + h f(x_0, y_0) = 1 + (0.05)\left[0^2 + 1^2\right] = 1.05$

$$x_1 = x_0 + h = 0 + 0.05 = 0.05$$

$$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}\right]$$

$$\Rightarrow y_1 = 1 + (0.05)\left[\frac{0^2 + 1^2 + (0.05)^2 + (1.05)^2}{2}\right] = 1.0526$$

$\underline{n=1}$: $\quad y_2^* = 1.0526 + (0.05)\left[(0.5)^2 + (1.0526)^2\right] = 1.1082$

$$x_2 = 0.1; \quad y_2 = 1.0526 + (0.05)\left[\frac{(0.05)^2 + (1.0526)^2 + (0.1)^2 + (1.1082)^2}{2}\right]$$

$$\Rightarrow y_2 = 1.1113$$

$n=2$: $y_3^* = 1.1113 + (0.05)\left[(0.1)^2 + (1.1113)^2\right] = 1.1736$

$x_3 = 0.15$; $y_3 = 1.1113 + (0.05)\left[\dfrac{(0.1)^2 + (1.1113)^2 + (0.15)^2 + (1.1736)^2}{2}\right]$

$\Rightarrow y_3 = 1.1775$

$n=3$: $y_4^* = 1.1775 + (0.05)\left[(0.15)^2 + (1.1775)^2\right] = 1.2479$

$x_4 = 0.2$; $y_4 = 1.1775 + (0.05)\left[\dfrac{(0.15)^2 + (1.1775)^2 + (0.2)^2 + (1.2479)^2}{2}\right]$

$\Rightarrow y_4 = 1.2526$

$n=4$: $y_5^* = 1.2526 + (0.05)\left[(0.2)^2 + (1.2526)^2\right] = 1.3331$

$x_5 = 0.25$; $y_5 = 1.2526 + (0.05)\left[\dfrac{(0.2)^2 + (1.2526)^2 + (0.25)^2 + (1.3331)^2}{2}\right]$

$\Rightarrow y_5 = 1.3388$

$n=5$: $y_6^* = 1.3388 + (0.05)\left[(0.25)^2 + (1.3388)^2\right] = 1.4316$

$x_6 = 0.3$; $y_6 = 1.3388 + (0.05)\left[\dfrac{(0.25)^2 + (1.3388)^2 + (0.3)^2 + (1.4316)^2}{2}\right]$

$\Rightarrow y_6 = 1.4387$

$n=6$: $y_7^* = 1.4387 + (0.05)\left[(0.3)^2 + (1.4387)^2\right] = 1.5467$

$x_7 = 0.35$; $y_7 = 1.4387 + (0.05)\left[\dfrac{(0.3)^2 + (1.4387)^2 + (0.35)^2 + (1.5467)^2}{2}\right]$

$\Rightarrow y_7 = 1.5556$

$n=7$: $y_8^* = 1.5556 + (0.05)\left[(0.35)^2 + (1.5556)^2\right] = 1.6827$

$x_8 = 0.4$; $y_8 = 1.5556 + (0.05)\left[\dfrac{(0.35)^2 + (1.5556)^2 + (0.4)^2 + (1.6827)^2}{2}\right]$

$\Rightarrow y_8 = 1.6939$

$\underline{n=8}$: $y_9^* = 1.6939 + (0.05)\left[(0.4)^2 + (1.6939)^2\right] = 1.8454$

$x_9 = 0.45$; $y_9 = 1.6939 + (0.05)\left[\frac{(0.4)^2 + (1.6939)^2 + (0.45)^2 + (1.8454)^2}{2}\right]$

$\Rightarrow y_9 = 1.8598$

$\underline{n=9}$: $y_{10}^* = 1.8598 + (0.05)\left[(0.45)^2 + (1.8598)^2\right] = 2.0429$

$x_{10} = 0.5$; $y_{10} = 1.8598 + (0.05)\left[\frac{(0.45)^2 + (1.8598)^2 + (0.5)^2 + (2.0429)^2}{2}\right]$

$\Rightarrow y_{10} = 2.0619$

EXERCISE : 6.2

Use RK4 method with $h = 0.1$ to obtain a four decimal approximation to the indicated value:

(4) $y' = 4x - 2y$; $y(0) = 2$; $y(0.5)$

$\underline{\text{sol}}$:- Given that $y(0) = 2$

$\Rightarrow x_0 = 0$ & $y_0 = 2$

$f(x_n, y_n) = 4x_n - 2y_n$; $h = 0.1$

$\underline{n=0}$: $y_1 = y_0 + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$; $x_1 = 0.1$

$k_1 = f(x_0, y_0) = 4x_0 - 2y_0 = 4(0) - 2(2) = -4$

$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_1) = 4\left[0 + \frac{1}{2}(0.1)\right] - 2\left[2 + \frac{1}{2}(0.1)(-4)\right]$

$\Rightarrow k_2 = -3.4000$

$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_2) = 4\left[0 + \frac{1}{2}(0.1)\right] - 2\left[2 + \frac{1}{2}(0.1)(-3.4)\right]$

$\Rightarrow k_3 = -3.4600$

$k_4 = f(x_0 + h, y_0 + hk_3) = 4\left[0 + 0.1\right] - 2\left[2 + (0.1)(-3.4600)\right] = -2.908$

$\therefore y_1 = 2 + \frac{0.1}{6}\left[-4 + 2(-3.4) + 2(-3.46) - 2.908\right]$

$\Rightarrow y_1 = 1.6562$

$\underline{n=1:}$ $\quad K_1 = 4(0.1) - 2(1.6562) = -2.9124$

$x_2 = 0.2$ $\quad K_2 = 4\left[(0.1) + \left(\frac{0.1}{2}\right)\right] - 2\left[1.6562 + \frac{(0.1)(-2.9124)}{2}\right]$

$\Rightarrow K_2 = -2.4212$

$K_3 = 4\left[(0.1) + \left(\frac{0.1}{2}\right)\right] - 2\left[1.6562 + \frac{(0.1)(-2.4212)}{2}\right]$

$\Rightarrow K_3 = -2.4703$

$K_4 = 4\left[0.1 + 0.1\right] - 2\left[1.6562 + (0.1)(-2.4703)\right]$

$\Rightarrow K_4 = -2.0183$

$\therefore y_2 = 1.6562 + \frac{0.1}{6}\left[-2.9124 + 2(-2.4212) + 2(-2.4703) - 2.0183\right]$

$\Rightarrow y_2 = 1.4110$

$\underline{n=2:}$ $\quad x_3 = 0.3$

$K_1 = 4(0.2) - 2(1.4110) = -2.022$

$K_2 = 4\left[0.2 + \frac{0.1}{2}\right] - 2\left[1.4110 + \frac{(0.1)}{2}(-2.022)\right] = -1.6198$

$K_3 = 4\left[0.2 + \frac{0.1}{2}\right] - 2\left[1.4110 + \frac{(0.1)}{2}(-1.6198)\right] = -1.6600$

$K_4 = 4\left[0.2 + 0.1\right] - 2\left[1.4110 + (0.1)(-1.6600)\right] = -1.29$

$\therefore y_3 = 1.4110 + \frac{0.1}{6}\left[-2.022 + 2(-1.6198) + 2(-1.6600) - 1.29\right]$

$\Rightarrow y_3 = 1.2465$

$\underline{n=3:}$ $\quad x_4 = 0.4$

$K_1 = 4(0.3) - 2(1.2465) = -1.293$

$$K_2 = 4\left[0.3 + \frac{0.1}{2}\right] - 2\left[1.2465 + \frac{(0.1)}{2}(-1.293)\right] = -0.9637$$

$$K_3 = 4\left[0.3 + \frac{0.1}{2}\right] - 2\left[1.2465 + \frac{(0.1)}{2}(-0.9637)\right] = -0.9966$$

$$K_4 = 4\left[0.3 + 0.1\right] - 2\left[1.2465 + (0.1)(-0.9966)\right] = -0.6937$$

$$\therefore y_4 = 1.2465 + \frac{0.1}{6}\left[-1.293 + 2(-0.9637) + 2(-0.9966) - 0.6937\right]$$

$$\Rightarrow y_4 = 1.1480$$

$\underline{n = 5:}$ $\quad x_5 = 0.5$

$$K_1 = 4(0.4) - 2(1.1480) = -0.696$$

$$K_2 = 4\left[0.4 + \frac{0.1}{2}\right] - 2\left[1.1480 + \frac{0.1}{2}(-0.696)\right] = -0.4264$$

$$K_3 = 4\left[0.4 + \frac{0.1}{2}\right] - 2\left[1.1480 + \frac{0.1}{2}(-0.4264)\right] = -0.4534$$

$$K_4 = 4(0.4 + 0.1) - 2\left[1.1480 + (0.1)(-0.4534)\right] = -0.2053$$

$$y_5 = 1.1480 + \frac{0.1}{6}\left[-0.696 + 2(-0.4264) + 2(-0.4534) - 0.2053\right]$$

$$\Rightarrow y_5 = 1.1037$$

EXERCISE : 6.4

(a) Use Euler's method to approximate $y(1.2)$, where $y(x)$ is the solution of Initial Value Problem

$$x^2 y'' - 2xy' + 2y = 0 \; ; \; y(1) = 4 \; ; \; y'(1) = 9$$

where $x > 0$. Use $h = 0.1$. Find the analytic Sol^n of the problem, and compare the actual value of $y(1.2)$ with y_2.

Sol:- Given that

$$x^2y'' - 2xy' + 2y = 0 \; ; \quad y(1) = 4 \; ; \; y'(1) = 9$$
$\hookrightarrow$ (1) $\qquad h = 0.1$

Let $y' = u \Rightarrow y'' = u'$

$\therefore$ from (1), $x^2 u' = 2xu - 2y$

$$\Rightarrow u' = \frac{2u}{x} - 2(y/x^2)$$

$$y_{n+1} = y_n + h\,u_n$$

and $U_{n+1} = U_n + h\left[\frac{2}{x}u_n - \frac{2}{x^2}y_n\right]$

$x_0 = 1; \quad y_0 = 4; \quad U_0 = 9$

$x_1 = 1.1; \; y_1 = y_0 + 0.1\,u_0 = 4 + (0.1)(9) = 4.9$

$U_1 = U_0 + 0.1\left[\frac{2}{1}U_0 - \frac{2}{1}y_0\right] = 9 + (0.1)\left[2(9) - 2(4)\right] = 10$

$y_2 = y_1 + 0.1\,U_1 = 4.9 + 0.1(10) = 5.9$

and $x_2 = 1.2$

$\therefore \boxed{y(1.2) = 5.9}$

From (1), $\quad y = x^m \Rightarrow y' = mx^{m-1} \; ; \; y'' = m(m-1)x^{m-2}$

$\Rightarrow m(m-1)x^{m-2}\cdot x^2 - 2x\,m x^{m-1} + 2x^m = 0$

$\Rightarrow x^m\left[m^2 - 3m + 2\right] = 0$

$x^m \neq 0 \Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$

$\Rightarrow y = C_1 x + C_2 x^2$ — (a)

Given that

$y(1) = 4$

$\Rightarrow$ From (a)

$c_1 + c_2 = 4$

and $y'(1) = 9$

$y'(x) = c_1 + 2c_2 x$

$\Rightarrow c_1 + 2c_2 = 9$

on solving, $c_1 = -1$; $c_2 = 5$

$\therefore y = -x + 5x^2$

$y(1.2) = -1.2 + 5(1.2)^2$

$\Rightarrow \boxed{y(1.2) = 6}$

EX: 6.4

(4) Repeat the above problem using "RK4" method. First use $h = 0.2$ & then use $h = 0.1$

Sol:- $x^2 y'' - 2xy' + 2y = 0$ — (1) $y(1) = 4$; $y'(1) = 9$

$y(1) = 4 \Rightarrow x_0 = 1$ & $y_0 = 4$

~~let $y' = t \Rightarrow y'(1) = 9 \Rightarrow t = t_0 = 9$~~

From (1) $y'' = \frac{2y'}{x} - 2(y/x^2)$

Let $y' = z \Rightarrow z' = y''$

$\Rightarrow z' = \frac{2z}{x} - 2(y/x^2)$ — (I)

From (I) $f(x, y, z) = z$

$g(x, y, z) = \frac{2z}{x} - 2(y/x^2)$

For $h = 0.2$ $\qquad x_0 = 1;\ y_0 = 4;\ z_0 = 9$

$$K_1 = hf[x_0, y_0, z_0]$$

$$\Rightarrow K_1 = (0.2)[9] = 1.8$$

$$l_1 = hg[x_0, y_0, z_0] = 0.2\left[\frac{2}{1}(9) - \frac{2}{1}(4)\right] = 0.2(10) = 2$$

$$K_2 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}K_1,\ z_0 + \tfrac{1}{2}l_1)$$

$$\Rightarrow K_2 = (0.2)f[(1+0.1), (4+0.9), (9+1)]$$

$$K_2 = (0.2)[10]$$

$$\Rightarrow K_2 = 2$$

$$l_2 = hg[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}K_1,\ z_0 + \tfrac{1}{2}l_1]$$

$$= (0.2)g[1.1,\ 4.9,\ 10] = (0.2)\left[\frac{2}{1.1}(10) - \frac{2}{(1.1)^2}(4.9)\right]$$

$$\Rightarrow l_2 = 2.0165$$

$$K_3 = hf[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}K_2,\ z_0 + \tfrac{1}{2}l_2]$$

$$= (0.2)f[1.1,\ \{4 + \tfrac{1}{2}(2)\},\ \{9 + \tfrac{1}{2}(2.0165)\}]$$

$$= 0.2[10.0083]$$

$$\Rightarrow K_3 = 2.0017$$

$$l_3 = hg[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}K_2,\ z_0 + \tfrac{1}{2}l_2] = (0.2)g[1.1, 5, 10.0083]$$

$$= (0.2)\left[\frac{2}{1.1}(10.0083) - \frac{2}{(1.1)^2}(5)\right]$$

$$\Rightarrow l_3 = 1.9865$$

$$K_4 = hf[x_0+h,\ y_0+K_3,\ z_0+l_3]$$

$$= (0.2)f[1.2,\ (4+2.0017),\ (9+1.9865)]$$

$$= (0.2)[10.9865]$$

$$\Rightarrow K_4 = 2.1973$$

$$l_4 = hg[x_0+h,\ y_0+K_3,\ z_0+l_3]$$

$$= (0.2)g[1.2,\ 6.0017,\ 10.9865]$$

$$= (0.2)\left[\frac{2}{1.2}(10.9865) - \frac{2}{(1.2)^2}(6.0017)\right]$$

$$\Rightarrow l_4 = 1.9950$$

$$\therefore y_1 = y_0 + \frac{1}{6}\{K_1 + 2K_2 + 2K_3 + K_4\}$$

$$\Rightarrow y_1 = 4 + \frac{1}{6}\{1.8 + 4 + 4.0034 + 2.1973\}$$

$$\Rightarrow y_1 = 6.0001$$

$$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 9 + \frac{1}{6}[2 + 4.033 + 3.973 + 1.995]$$

$$\Rightarrow z_1 = 11.0002$$

at $x_1 = 1.2$

For $h = 0.1$:

$$x_0 = 1;\ y_0 = 4;\ z_0 = 9$$

$$\Rightarrow x_1 = 1 + 0.1 = 1.1$$

$$K_1 = hf(x_0, y_0, z_0) = (0.1)[z_0] = (0.1)9 = 0.9$$

$$l_1 = hg(x_0, y_0, z_0) = (0.1)\left[\frac{2}{1}(9) - \frac{2}{1}(4)\right] = 1$$

$$K_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{K_1}{2},\ z_0 + \tfrac{l_1}{2}\right)$$

$$= (0.1)\ f\left[\left(1 + \tfrac{0.1}{2}\right), \left(4 + \tfrac{0.9}{2}\right), \left(9 + \tfrac{1}{2}\right)\right] = (0.1)(9.5)$$

$$\Rightarrow K_2 = 0.95$$

$$l_2 = hg\left[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{K_1}{2},\ z_0 + \tfrac{l_1}{2}\right] = (0.1)g\left[1.05,\ 4.45,\ 9.5\right]$$

$$= (0.1)\left[\frac{2}{1.05}(9.5) - \frac{2}{(1.05)^2}(4.45)\right]$$

$$\Rightarrow l_2 = 1.0023$$

$$K_3 = hf\left[x_0 + \tfrac{h}{2},\ y_0 + \tfrac{K_2}{2},\ z_0 + \tfrac{l_2}{2}\right]$$

$$= (0.1)\ f\left[1.05,\ \left(4 + \tfrac{0.95}{2}\right),\ \left(9 + \tfrac{1.0023}{2}\right)\right]$$

$$= (0.1)\left[9.5012\right]$$

$$\Rightarrow K_3 = 0.95012$$

$$l_3 = hg\left[x_0 + \tfrac{h}{2},\ y_0 + \tfrac{K_2}{2},\ z_0 + \tfrac{l_2}{2}\right]$$

$$= (0.1)\left[\frac{2}{1.05}(9.5012) - \frac{2}{(1.05)^2}(4.475)\right]$$

$$\Rightarrow l_3 = 0.99796$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3,\ z_0 + l_3)$$

$$= (0.1)\ f\left[1.1,\ (4 + 0.95012),\ (9 + 0.99796)\right]$$

$$= (0.1)\left[9.99796\right]$$

$$\Rightarrow K_4 = 0.999796$$

$$l_4 = (0.1)g\left[1.1,\ 4.95012,\ 9.99796\right]$$

$$= (0.1)\left[\frac{2}{1.1}(9.99796) - \frac{2}{(1.1)^2}(4.95012)\right]$$

$$\Rightarrow l_4 = 0.99961$$

$\therefore y_1 = 4 + \frac{1}{6}[0.9 + 1.9 + 1.90024 + 0.999796]$

$\Rightarrow y_1 = 4.9500$

$z_1 = 9 + \frac{1}{6}\{1 + 2.0046 + 1.99592 + 0.99961]$

$\Rightarrow z_1 = 10.00002$

at $x_1 = 1.1$

for $x_2 = 1.2$

$y_1 = 4.9500$; $z_1 = 10.0000$

$\therefore k_1 = 0.1\, f[1.1, 4.95, 10.0000] = 1$

$l_1 = 0.1 \left[\frac{2}{1.1}(10) - \frac{2}{(1.1)^2}(4.95)\right] = 1$

$k_2 = (0.1) f[1.15, 5.45, 10.5] = 0.1(10.5)$

$\Rightarrow k_2 \; 1.05$

$l_2 = (0.1)\left[\frac{2}{1.15}(10.5) - \frac{2}{(1.15)^2}(5.45)\right]$

$\Rightarrow l_2 = 1.00189$

$k_3 = (0.1) f\{1.15, 5.475, 10.5009] = (0.1)(10.5009)$

$\Rightarrow k_3 = 1.05009$

$l_3 = (0.1)\left[\frac{2}{1.15}(10.5009) - \frac{2}{(1.15)^2}(5.475)\right]$

$\Rightarrow l_3 = 0.99826$

$k_4 = (0.1) f[1.2, 6.00009, 10.99826] = (0.1)[10.99826]$

$\Rightarrow k_4 = 1.09982$

$l_4 = (0.1)\left[\frac{2}{1.2}(10.99826) - \frac{2}{(1.2)^2}(6.00009)\right] = 0.9997$

$$\therefore\ y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 4.95 + \frac{1}{6}\left[1 + 2.1 + 2.10018 + 1.09998\right]$$

$$\Rightarrow y_2 = 6.0000$$

$$z_2 = z_1 + \frac{1}{6}\left\{l_1 + 2l_2 + 2l_3 + l_4\right]$$

$$= 10.0000 + \frac{1}{6}\left\{1 + 2.00378 + 1.99652 + 0.9997\right]$$

$$\Rightarrow z_2 = 11.0000$$

at $x_2 = 1.2$