SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 06

DUE ON: 07/23/'08

QUESTIONS

(1) Find the directional derivative of given function at the given Point in the given direction.

(a) $f(x,y) = x^2 + xy + y^2$; $(1,2)$; $\theta = \frac{\pi}{3}$

(b) $f(x,y) = x^2 - 3yz$ in the direction of $\bar{i} + \bar{j} - 2\bar{k}$ at $[2,-1,4]$

(2) Find a vector that gives the direction in which the given function increases most rapidly at the indicated Point. Find the max. rate: $F(x,y,z) = 3x^2y - xz$ at $(1,2,5)$.

(3) Find the equation of tangent plane to:

(a) $xy^2z^3 = 12$ at $(3,2,1)$.

(b) $x^2 + y^2 + z = 9$ at $(1,1,7)$

(4) Find the curl and divergence of given vector field.

(a) $\vec{F}(x,y,z) = x^2y\,\bar{i} + (y^2 - x^2)\,\bar{j} + xy\,\bar{k}$

(b) $\vec{F}(x,y,z) = x^3y^2z\,\bar{i} + x^2z\,\bar{j} + x^2y\,\bar{k}$

(5) Evaluate $\int_C y\,dx + x\,dy + z\,dz$ where 'C' is given by

$x = \cos t$; $y = \sin t$; $z = t^2$; $0 \leq t \leq 2\pi$.

(6) Find the work done by the force $\vec{F} = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$ over the straight line from (1,1) to (2,3).

(7) Find the work done by the constant force $\vec{F}(x,y) = 3\vec{i} + 2\vec{j}$ acting counter clock wise once around the circle $x^2 + y^2 = 16$

SOLUTIONS :

(1) Find the directional derivative of given function at given point in the given direction.

(a) $f(x,y) = x^2 + xy + y^2$; $(1,2)$; $\theta = \frac{\pi}{3}$.

Sol:- $f(x,y) = x^2 + xy + y^2$

$$\nabla f = (2x+y)\,\bar{i} + (x+2y)\,\bar{j}$$

$$\Rightarrow \nabla f(1,2) = 4\bar{i} + 5\bar{j}$$

Given $\theta = \frac{\pi}{3} \Rightarrow u = \cos\left(\frac{\pi}{3}\right)\bar{i} + \sin\left(\frac{\pi}{3}\right)\bar{j}$

$$\Rightarrow \hat{u} = \frac{1}{2}\bar{i} + \frac{\sqrt{3}}{2}\bar{j}$$

Now, directional derivative,

$$D_u f = \nabla f \cdot \hat{u} = (4\bar{i} + 5\bar{j}) \cdot \left(\frac{1}{2}\bar{i} + \frac{\sqrt{3}}{2}\bar{j}\right)$$

$$\Rightarrow D_u f = 2 + \frac{5\sqrt{3}}{2}$$

(b) $f(x,y,z) = x^2 - 3yz$ in the direction of $\bar{i} + \bar{j} - 2\bar{k}$ at $[2,-1,4]$.

Sol:- $f(x,y,z) = x^2 - 3yz$

$$\therefore \nabla f = (2x)\bar{i} - (3z)\bar{j} - (3y)\bar{k}$$

$$\nabla f(2,-1,4) = 4\bar{i} - 12\bar{j} + 3\bar{k}$$

Given $\bar{u} = \bar{i} + \bar{j} - 2\bar{k}$ which is not a unit vector.

$$\therefore \hat{u} = \frac{\bar{u}}{\|\bar{u}\|} = \frac{(\bar{i}+\bar{j}-2\bar{k})}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}\left[\bar{i}+\bar{j}-2\bar{k}\right]$$

Now, $D_u f = \nabla f \cdot \hat{u} = (4\bar{i} - 12\bar{j} + 3\bar{k}) \cdot \left(\frac{\bar{i}+\bar{j}-2\bar{k}}{\sqrt{6}}\right)$

$$= \frac{4 - 12 - 6}{\sqrt{6}}$$

$$\Rightarrow D_u f = \frac{-14}{\sqrt{6}}$$

(2) Find a vector that gives the direction in which the given function increases most rapidly at the indicated point. Find the max. rate: $F(x,y,z) = 3x^2y - xz$ at $(1,2,5)$

Sol:- Given $F(x,y,z) = 3x^2y - xz$

$$\Rightarrow \nabla F = (6xy - z)\bar{i} + (3x^2)\bar{j} - (x)\bar{k}$$

$$\nabla F(1,2,5) = 7\bar{i} + 3\bar{j} - \bar{k}$$

Max. D_u is $\sqrt{7^2 + 3^2 + (-1)^2} = \sqrt{59}$

in the direction of $7\bar{i} + 3\bar{j} - \bar{k}$

(3) Find the equation of tangent plane to:

(a) $xy^2z^3 = 12$ at $(3,2,1)$

Sol:- let $f(x,y,z) = xy^2z^3$

$\Rightarrow \nabla f = (y^2z^3)\bar{i} + (2xyz^3)\bar{j} + (3xy^2z^2)\bar{k}$

$\nabla f(3,2,1) = 4\bar{i} + 12\bar{j} + 36\bar{k}$

The eq. of tangent plane is

$4(x-3) + 12(y-2) + 36(z-1) = 0$

$\Rightarrow 4x + 12y + 36z = 72$

$\Rightarrow x + 3y + 9z = 18$

(b) $x^2+y^2+z = 9$ at $(1,1,7)$

Sol:- let $f(x,y,z) = x^2+y^2+z$

$\Rightarrow \nabla f = (2x)\bar{i} + (2y)\bar{j} + \bar{k}$

$\nabla f(1,1,7) = 2\bar{i} + 2\bar{j} + \bar{k}$

The eq. of tangent plane is

$2(x-1) + 2(y-1) + (z-7) = 0$

$\Rightarrow 2x + 2y + z = 11$

(4) Find the curl and divergence of given vector field.

(a) $\vec{F}(x, y, z) = x^2 y\,\bar{i} + (y^2 - x^2)\,\bar{j} + xy\,\bar{k}$

Sol:- $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & y^2 - x^2 & xy \end{vmatrix}$

$$\Rightarrow \nabla \times \vec{F} = \bar{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(y^2 - x^2)\right] - \bar{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x^2 y)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(y^2 - x^2) - \frac{\partial}{\partial y}(x^2 y)\right]$$

$$= (x)\,\bar{i} - (y)\,\bar{j} + (-2x - x^2)\,\bar{k}$$

$$\Rightarrow \boxed{\nabla \times \vec{F} = (x)\,\bar{i} - (y)\,\bar{j} - (2x + x^2)\,\bar{k}}$$

$$\text{Div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y^2 - x^2) + \frac{\partial}{\partial z}(xy)$$

$$\Rightarrow \boxed{\nabla \cdot \vec{F} = 2xy + 2y}$$

(b) $\vec{F}(x, y, z) = x^3 y^2 z\,\bar{i} + x^2 z\,\bar{j} + x^2 y\,\bar{k}$

Sol:- $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 y^2 z & x^2 z & x^2 y \end{vmatrix}$

$$\Rightarrow \nabla \times \vec{F} = (x^2 - x^2)\bar{i} - (2xy - x^3y^2)\bar{j} + (2xz - 2x^3yz)\bar{k}$$

$$\Rightarrow \boxed{\nabla \times \vec{F} = (x^3y^2 - 2xy)\bar{j} + (2xz - 2x^3yz)\bar{k}}$$

$$\text{Div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3y^2z) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(x^2y)$$

$$\Rightarrow \boxed{\nabla \cdot \vec{F} = 3x^2y^2z}$$

(5) Evaluate $\int_c y\,dx + x\,dy + z\,dz$ where 'c' is given by

$x = \cos t$; $y = \sin t$; $z = t^2$; $0 \le t \le 2\pi$

Sol:- Given $\int_c y\,dx + x\,dy + z\,dz$

$x = \cos t \Rightarrow dx = -\sin t\,dt$

$y = \sin t \Rightarrow dy = \cos t\,dt$

$z = t^2 \Rightarrow dz = 2t\,dt$

$0 \le t \le 2\pi$.

$\therefore \int_c y\,dx + x\,dy + z\,dz$

$$= \int_0^{2\pi} [\sin t(-\sin t)\,dt] + [\cos t(\cos t)\,dt] + [(t^2)(2t)\,dt]$$

$$= \int_0^{2\pi} \left[(-\sin^2 t) + \cos^2 t + 2t^3\right] dt$$

$$= \int_0^{2\pi} (\cos 2t + 2t^3)\, dt$$

$$= \left[\frac{\sin 2t}{2} + \frac{t^4}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\sin 2(2\pi) + (2\pi)^4 - (\sin 2(0) - (0)^4)\right]$$

$$= \frac{1}{2}\left[0 + 16\pi^4\right] = 8\pi^4$$

$$\therefore \int_c x\,dy + y\,dx + z\,dz = 8\pi^4.$$

(6) Find the work done by the force

$$\vec{F} = (x^2 - yx)\bar{i} + (y^2 - xy)\bar{j}$$

over the straight line from (1,1) to (2,3).

Sol: Given $\vec{F} = (x^2 - yx)\bar{i} + (y^2 - xy)\bar{j}$

straight line between (1,1) & (2,3) is $y = 2x - 1$

$$\left[(y-1) = \frac{3-1}{2-1}(x-1)\right]$$
$$\{\Rightarrow y = 2x - 1\}$$

with $y = 2x - 1$

$dy = 2\,dx$

Now, work done, $W = \int_c F \cdot dr$

where $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\therefore \vec{F} \cdot d\vec{r} = (x^2 - yx)\,dx + (y^2 - xy)\,dy$

with $y = 2x - 1$ & $dy = 2dx$

$$\int_c \bar{F} \cdot d\bar{r} = \int_1^2 \left[x^2 - (2x-1)x\right] dx + \left\{(2x-1)^2 - x(2x-1)\right\} 2dx$$

$$= \int_1^2 (x^2 - 2x^2 + x)\,dx + 2(2x-1)\left[(2x-1) - x\right] dx$$

$$= \int_1^2 (-x^2 + x)\,dx + (4x^2 - 6x + 2)\,dx$$

$$= \int_1^2 (3x^2 - 5x + 2)\,dx$$

$$= \left. x^3 - \frac{5x^2}{2} + 2x \right]_1^2$$

$$= \left[8 - \frac{20}{2} + 4\right] - \left[1 - \frac{5}{2} + 2\right]$$

$$= 2 - \frac{1}{2}$$

$$\Rightarrow \int_c F \cdot dr = \frac{3}{2}$$

$\therefore$ Work done $= \frac{3}{2}$ units.

Alternative Method: (Parameterisation)

let $\left.\begin{aligned} x &= t \\ y &= 2t - 1 \end{aligned}\right\}$ $1 \le t \le 2$

$\Rightarrow \bar{r} = t\bar{i} + (2t-1)\bar{j}$

$\therefore d\bar{r} = dt\,\bar{i} + 2dt\,\bar{j}$

Now, the work done can be found out using $\int_c \bar{F}\cdot d\bar{r}$

(7) Find the work done by the constant force $\vec{F}(x,y) = 3\bar{i} + 2\bar{j}$ acting counter clock wise direction once around the circle $x^2 + y^2 = 16$.

Sol:- Given $\vec{F}(x,y) = 3\bar{i} + 2\bar{j}$

$$x^2 + y^2 = 16$$

$$\therefore a = 4\cos t;\ b = 4\sin t$$

$$\bar{r} = (4\cos t,\ 4\sin t)$$

(or) $\bar{r} = (4\cos t)\bar{i} + (4\sin t)\bar{j} \quad ; \ 0 \le t \le 2\pi$

$$\therefore d\bar{r} = \left[(-4\sin t)\bar{i} + (4\cos t)\bar{j}\right]dt$$

$$\text{Work done, } W = \int_c F\cdot dr = \int_0^{2\pi}\left[[-4\sin t](3) + [4\cos t](2)\right]dt$$

$$= \int_0^{2\pi} (-12\sin t + 8\cos t)\,dt$$

$$\Rightarrow W = \left[12\cos t + 8\sin t\right]_0^{2\pi} = 12 - 12 = 0$$

$$\therefore \boxed{\text{Work done} = 0}$$