MAE 3360 HW #2 ①

ENGINEERING ANALYSIS

Exercises 2.2 – Separable Variables

6. $\frac{dy}{dx} + 2xy^2 = 0$

$\frac{dy}{dx} = -2xy^2$

$\frac{1}{y^2} dy = -2x\,dx$

Integrating both sides,

$\int \frac{1}{y^2} dy = \int -2x\,dx$

$-\frac{1}{y} = -x^2 + c$

$\therefore \boxed{y = \frac{1}{x^2 + c_1}}$

8. $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$

$\therefore e^y y \frac{dy}{dx} = e^{-x} + e^{-3x}$

$\therefore e^y \cdot y\, dy = (e^{-x} + e^{-3x})\,dx$

Integrating both sides,

$$\int ye^{y}\,dy = \int e^{-x}dx + \int e^{-3x}dx$$

$$\therefore ye^{y} - e^{y} = -e^{-x} - \frac{e^{-3x}}{3} + c$$

$$\therefore \boxed{e^{y}(y-1) = -e^{-x} - \frac{e^{-3x}}{3} + c}$$

Exercises 2-5 — Solutions by Substitutions

5. $(y^2 + yx)\,dx - x^2 dy = 0$

Put $y = ux \quad \therefore dy = u\,dx + x\,du$

$$(u^2x^2 + ux^2)\,dx - x^2(u\,dx + x\,du) = 0$$

$$\therefore u^2x^2\,dx - x^3 du = 0$$

$$\therefore -\frac{dx}{x} = -\frac{dv}{v2}$$

Integrating both sides,

$$-\int \frac{dx}{x} = -\int \frac{dv}{v2}$$

$$-\ln x - \frac{1}{v} = c$$

$$\therefore \ln x = c - \frac{x}{y}$$

$$\therefore \boxed{y \ln x + x = cy}$$

11. $$xy^2 \frac{dy}{dx} = y^3 - x^3$$

$$xy^2\, dy + (x^3 - y^3)\, dx = 0$$

Put $y = ux$ $\therefore dy = u\,dx + x\,du$

$$x^3 u^2 dy + (x^3 - u^3 x^3) dx = 0$$

$$\therefore x^3 u^2 dy + x^3 dx - u^3 x^3 dx = 0$$

$$x^3 u^3 dx + x^4 u^2 du + x^3 dx - v^3 x^3 dx = 0$$

$$\therefore x^4 u^2 du = -x^3 dx$$

$$\therefore u^2 du = -\frac{1}{x} dx$$

Integrating both sides,

$$\frac{u^3}{3} = -\ln x + c$$

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Put $v = \frac{y}{x}$

$$\boxed{\therefore y^3 = -3x^3 \ln x + 3x^3 c}$$

when $y(1) = 2$

$$2^3 = -3(1)^3 \ln 1 + 3(1)^3 c$$

$$\therefore c = \frac{8}{3}$$

$$\therefore \boxed{y^3 + 3x^3 \ln x = 8x^3}$$

17. $\frac{dy}{dx} = y(xy^3 - 1)$

$$\frac{dy}{dx} = xy^4 - y$$

$$\therefore \frac{dy}{dx} + y = xy^4$$

$$u = y^{1-n} = y^{1-4} = y^{-3}$$

$$\therefore \frac{du}{dx} = -3y^{-4} \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{3} y^4 \frac{du}{dx}$$

$$\therefore \frac{-1}{3} y^4 \frac{du}{dx} + y = xy^4$$

$$\therefore \frac{du}{dx} - 3y^{-3} = -3x$$

$$\therefore \frac{du}{dx} - 3u = -3x$$

Hence, integrating factor is $e^{-\int 3\, dx} = e^{-3x}$

$$e^{-3x} \frac{du}{dx} - 3ue^{-3x} = -3e^{-3x} \cdot x$$

$$\therefore \frac{d}{dx}\left(u \cdot e^{-3x}\right) = -3xe^{-3x}$$

Integrating both sides,

$$\int \frac{d}{dx}(ue^{-3x})\, dx = -3\int xe^{-3x}\, dx$$

$$\therefore ue^{-3x} = xe^{-3x} + \frac{1}{3} e^{-3x} + c\cancel{e^{-3x}}$$

$$\therefore u = x + \frac{1}{3} + ce^{3x}$$

Put $u = y^{-3}$

$$\boxed{\therefore\ y = x + \frac{1}{3} + ce^{3x}}$$

2) $x^2\frac{dy}{dx} - 2xy = 3y^4$

$\therefore \frac{dy}{dx} - \frac{2}{x}y = \frac{3y^4}{x^2}$

Let $u = y^{-3}$ $\quad \therefore \frac{du}{dy} = -3y^{-4}\frac{dy}{dx}$

$\therefore -\frac{1}{3}\frac{du}{dx} - \frac{2}{x}u = 3x^{-2}$ $\quad \because u = y^{-3}$

$\therefore \frac{du}{dx} + \frac{6}{x}u = -9x^{-2}$

Integrating factor is $e^{6\int\frac{dx}{x}} = e^{6\ln x} = x^6$

$\therefore x^6\frac{du}{dx} + \frac{6}{x}\cdot x^6\cdot u = -9x^{-2}\cdot x^6$

$\therefore \frac{d}{dx}(x^6u) = -9x^4$

Integrating both sides,

$x^6u = -\frac{9}{5}x^5 + c$

Put $u = y^{-3}$

$\therefore y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$

Given $y(1) = \frac{1}{2}$

$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1) + c$

$\therefore c = \frac{49}{5}$

$$\boxed{\therefore y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}}$$

Exercises 2.7 – Linear Models

4 Given: $\frac{dP}{dt} \propto P \quad \therefore \frac{dP}{dt} = kP$

$\therefore \frac{dP}{P} = k\,dt$

Integrating both sides

$\int \frac{dP}{P} = k\int dt$

$\therefore \ln P = kt + c$

$\therefore P = e^{kt+c} = e^{kt} \cdot e^{c}$

$\therefore P = P_0 e^{kt} \quad \ldots \because e^{c}$ is constant

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From the two given conditions,

$$400 = P_0 e^{3k}$$

$$\& \ 2000 = P_0 e^{10k}$$

Solving the above two equations, we get,

$$\boxed{P = 200.67 \sim 201 \text{ bacteria}}$$