SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 02

DUE ON: 02/06/08

QUESTIONS:

EXERCISE: 2.2

Solve the given d.e by separation of variables.

(5) $xy' = 4y$ (11) $\csc y\, dx + \sec^2 x\, dy = 0$

(19) $$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

EXERCISE: 2.5

Solve the given homogeneous d.e by using an appropriate substitution.

(5) $(y^2 + yx)dx - x^2 dy = 0$

(11) Solve the initial value problem $(xy^2)\dfrac{dy}{dx} = y^3 - x^3$; $y(1) = 2$

Solve the given d.e (Bernoulli eq.) by using an appropriate substitution.

(17) $y' = y(xy^3 - 1)$ (19) $t^2 \dfrac{dy}{dt} + y^2 = ty$

(22) Solve the given initial value problem

$$y^{1/2}\frac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$$

(23) Solve the given d.e by using an appropriate substitution.

$$\frac{dy}{dx} = (x + y + 1)^2$$

SOLUTIONS

EXERCISE : 2.2

Solve the given d.e. by separation of variables.

(5) $x \dfrac{dy}{dx} = 4y$

Sol:- Given $x \dfrac{dy}{dx} = 4y$

$$\Rightarrow \quad \frac{1}{y}\,dy = \frac{4}{x}\,dx$$

Integrating both sides, we get

$$\ln|y| = 4\ln|x| + c$$

$$\Rightarrow \quad y = C_1 x^4$$

(11) $\csc y\, dx + \sec^2 x\, dy = 0$

Sol:- Given $\csc y\, dx + \sec^2 x\, dy = 0$

$$\Rightarrow \quad \csc y\, dx = -\sec^2 x\, dy$$

$$\Rightarrow \quad -\frac{dx}{\sec^2 x} = \frac{1}{\csc y}\,dy$$

$$\Rightarrow \quad -\cos^2 x\, dx = \sin y\, dy$$

$$\Rightarrow \quad -\left[\frac{1 + \cos 2x}{2}\right] dx = \sin y\, dy$$

on Integration, we get

$$\frac{-x}{2} - \frac{1}{4}\sin 2x = -\cos y + c$$

$$\Rightarrow \quad 4\cos y = 2x + \sin 2x + c_1$$

Ex: 2.2

(19) $$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

Sol:- Given $$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(y+3) - 1(y+3)}{x(y-2) + 4(y-2)} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\Rightarrow \left(\frac{y-2}{y+3}\right) dy = \left(\frac{x-1}{x+4}\right) dx$$

$$\Rightarrow \left(\frac{y+3-5}{y+3}\right) dy = \left(\frac{x+4-5}{x+4}\right) dx$$

$$\Rightarrow \left(\frac{y+3}{y+3} - \frac{5}{y+3}\right) dy = \left(\frac{x+4}{x+4} - \frac{5}{x+4}\right) dx$$

$$\Rightarrow \left(1 - \frac{5}{y+3}\right) dy = \left(1 - \frac{5}{x+4}\right) dx$$

on integration we get,

$$y - 5 \ln|y+3| = x - 5 \ln|x+4| + c$$

$$\Rightarrow x - y = 5 \ln|x+4| - 5 \ln|y+3| + c$$

$$\Rightarrow \ln\left[\frac{x+4}{y+3}\right]^5 = (x-y) + c_1$$

$$\Rightarrow \left[\frac{x+4}{y+3}\right]^5 = e^{(x-y)+c_1} = e^{c_1} \cdot e^{(x-y)}$$

$$\Rightarrow \left[\frac{x+4}{y+3}\right]^5 = C_1 e^{(x-y)}$$

EXERCISE : 2.5

Solve the given homogeneous d.E by using an appropriate Substitution.

(5) $(y^2+yx)\,dx - x^2\,dy = 0$

Sol:- Let $y = ax \Rightarrow dy = a\,dx + x\,da$

On substitution,

$\therefore (a^2x^2 + ax^2)\,dx - x^2(a\,dx + x\,da) = 0$

$\Rightarrow (a^2x^2 + \cancel{ax^2} - \cancel{ax^2})\,dx - x^3\,da = 0$

$\Rightarrow a^2\cancel{x^2}\,dx = x^{\cancel{3}}\,da$

$\Rightarrow a^2\,dx = x\,da \quad \Rightarrow \dfrac{dx}{x} = \dfrac{da}{a^2}$

Integrating both sides,

$$\int \frac{dx}{x} = \int \frac{da}{a^2}$$

$\Rightarrow \ln|x| = \dfrac{-1}{a} + c$

$\Rightarrow \ln|x| + \dfrac{1}{(y/x)} = c \qquad (\because a = \dfrac{y}{x})$

$\Rightarrow y\ln|x| + x = cy$

(11) Solve the given initial-value problem:

$$(xy^2)\frac{dy}{dx} = y^3 - x^3 \ ; \ y(1) = 2$$

Sol:- Let $y = ax \Rightarrow dy = a\,dx + x\,da$

on substitution,

$(a^2x^3)(a\,dx + x\,da) = x^3(a^3-1)\,dx$

$$\Rightarrow \cancel{a^3x^3\,dx} + a^2x^4\,da = \cancel{a^3x^3\,dx} - x^3\,dx$$

$$\Rightarrow a^2 x^{\cancel{4}}\,da = -\cancel{x^3}\,dx$$

$$\Rightarrow a^2 x\,da = -dx$$

$$\Rightarrow a^2\,da = -\frac{dx}{x}$$

on Integration,

$$\int a^2\,da = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{a^3}{3} = -\ln|x| + c$$

$$\Rightarrow \frac{a^3}{3} + \ln|x| = c$$

$$\Rightarrow \ln|x| + \frac{(y/x)^3}{3} = c$$

$$\Rightarrow 3x^3\ln|x| + y^3 = 3cx^3 = c_1x^3$$

Given $y(1) = 2$

$\Rightarrow$ at $x = 1$, $y = 2$

$$\therefore 3(1)^3\ln|1| + 8 = c_1(1)^3$$

$$\Rightarrow c_1 = 8 \quad (\text{or}) \quad c = \frac{8}{3} \qquad (\because c_1 = 3c)$$

$$\therefore 3x^3\ln|x| + y^3 = 8x^3$$

Ex: 2.5

Solve the given d.e by using an appropriate substitution.

(Bernoulli's eq.)

(17) $y' = y(xy^3 - 1)$

Sol:- Given $y' = y(xy^3 - 1)$

$$\Rightarrow y' + y = xy^4$$

$$\Rightarrow (y^{-4})y' + \left(\frac{y}{y^4}\right) = x$$

$$\Rightarrow (y^{-4})y' + y^{-3} = x$$

let $a = y^{-3} \Rightarrow da = -3y^{-4}dy$

$$\Rightarrow y^{-4}dy = -\frac{da}{3}$$

$$\therefore (y^{-4})y' = \left(\frac{-1}{3}\right)a'$$

on Substitution,

$$\left(-\frac{1}{3}\right)a' + a = x$$

$$\Rightarrow a' - 3a = -3x$$

on comparision with standard form $a' + P(x)a = f(x)$

we have $P(x) = -3$; $f(x) = -3x$

$$\therefore If = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

$$\therefore \frac{d}{dx}\left[e^{-3x}a\right] = e^{-3x}(-3x)$$

on Integration,

$$ae^{-3x} = \int (-3x)e^{-3x}dx + c$$

let $-3x = t$

$\Rightarrow -3dx = dt$

$\Rightarrow dx = -\frac{dt}{3}$

$$\therefore ae^{t} = \int te^{t}\left(\frac{dt}{-3}\right) + c$$

$$\Rightarrow ae^{t} = \frac{-1}{3}\left[te^{t} - \int e^{t}dt\right] + c$$

$$\Rightarrow ae^{t} = \frac{-1}{3}e^{t}(t-1) + c$$

$$\Rightarrow a = \frac{-1}{3}(t-1) + ce^{-t}$$

$$\therefore y^{-3} = \frac{-1}{3}(-3x-1) + ce^{-(-3x)} = x + \frac{1}{3} + ce^{3x}$$

Ex: 2.5

(19) $t^2 \frac{dy}{dt} + y^2 = ty$

Sol: Given $t^2 y' + y^2 = ty$

$$\Rightarrow y' + \frac{y^2}{t^2} = \frac{y}{t}$$

$$\Rightarrow y' - \frac{y}{t} = -\frac{y^2}{t^2} \quad \text{— (a)}$$

Let $a = y^{1-2} = y^{-1}$

$$\Rightarrow y = 1/a \Rightarrow \frac{dy}{dt} = \frac{-1}{a^2} \frac{da}{dt}$$

$\therefore$ From (a),

$$\left(\frac{-1}{a^2}\right) a' - \frac{1}{at} = \frac{-1}{a^2 t^2}$$

$$\Rightarrow a' + \frac{a}{t} = \frac{1}{t^2}$$

On Compairision with standard form,

we have $P(t) = \frac{1}{t}$; $f(t) = \frac{1}{t^2}$

$$\therefore I.f = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$

$$\therefore \frac{d}{dt}[ta] = t\left[\frac{1}{t^2}\right]$$

On Integration,

$$at = \int \frac{1}{t} dt + c$$

$$\Rightarrow at = \ln|t| + c$$

$$\Rightarrow at = \ln|t| + \ln|c_1|$$

$$\Rightarrow \frac{t}{y} = \ln|t| + \ln|c_1| \qquad \left[\because a = 1/y \right.$$

$$\& \ \ln(ab) = x$$

$$\Rightarrow c_1 t = e^{(t/y)} \qquad \left. \Rightarrow ab = e^x \right]$$

Ex: 2.5

(22) Solve the given initial value problem.

$$y^{1/2} \frac{dy}{dx} + y^{3/2} = 1 \quad ; \ y(0) = 4.$$

Sol: Given $(y^{1/2}) y' + y^{3/2} = 1$

$$\Rightarrow y' + y = y^{-1/2} \quad \text{--- (i)}$$

$$\text{let } a = y^{1-(-1/2)} = y^{3/2} \qquad \Rightarrow \boxed{y = a^{2/3}}$$

$$\Rightarrow \frac{da}{dx} = \frac{3}{2} y^{1/2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{2}{3} y^{-1/2} \frac{da}{dx}\right] = \left[\frac{2}{3} (a^{2/3})^{-1/2} \frac{da}{dx}\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2}{3} a^{-1/3} \frac{da}{dx}}$$

$\therefore$ From eq. (i)

$$\left(\frac{2}{3} a^{-1/3} \frac{da}{dx}\right) + a^{2/3} = (a^{2/3})^{-1/2}$$

$$\Rightarrow \frac{2}{3} \frac{da}{dx} + a = 1$$

$$\Rightarrow \frac{da}{dx} + \frac{3}{2} a = \frac{3}{2}$$

on Comparing with standard form,

$$P(x) = 3/2 \ ; \ f(x) = 3/2$$

$$\therefore \quad I_f = e^{\int \frac{3}{2} dx} = e^{\frac{3}{2}x}$$

$$\therefore \quad \frac{d}{dx}\left[e^{\frac{3x}{2}} a\right] = \frac{3}{2}\left(e^{\frac{3}{2}x}\right)$$

on Integration,

$$a e^{\frac{3x}{2}} = \int \frac{3}{2} e^{(3/_2 x)} dx + c$$

$$\Rightarrow a e^{\frac{3x}{2}} = e^{\frac{3x}{2}} + c$$

$$\Rightarrow a = 1 + c e^{\left(-\frac{3x}{2}\right)}$$

$$\Rightarrow y^{3/2} = 1 + c e^{\left(-3x/_2\right)} \qquad \left(\because a = y^{3/2}\right)$$

Given $y(0) = 4$

$$\Rightarrow \text{at } x = 0, \; y = 4$$

$$\Rightarrow (4)^{3/2} = 1 + c$$

$$\Rightarrow c = 7$$

$$\therefore \quad y^{3/2} = 1 + 7e^{\left(\frac{-3x}{2}\right)}$$

Ex: 2.5

Solve the given d.e by using an appropriate substitution

(23) $y' = (x + y + 1)^2$

Sol:

let $x + y + 1 = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$\therefore$ on Substitution,

$$\frac{dt}{dx} - 1 = t^2$$

$$\Rightarrow \frac{dt}{dx} = t^2+1$$

$$\Rightarrow \frac{dt}{1+t^2} = dx$$

on Integration,

$$\int \frac{dt}{1+t^2} = \int dx$$

$$\Rightarrow \tan^{-1}(t) = x+c$$

$$\Rightarrow t = \tan(x+c)$$

$$\Rightarrow (x+y+1) = \tan(x+c) \qquad (\because t = x+y+1)$$

$$\Rightarrow y = \tan(x+c) - (x+1)$$