SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 03

DUE ON: 02/13/08

QUESTIONS

EXERCISE: 3.1

Determine whether the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

(17) $f_1(x) = 5$; $f_2(x) = \cos^2 x$; $f_3(x) = \sin^2 x$

(19) $f_1(x) = x$; $f_2(x) = x-1$; $f_3(x) = x+3$

Verify that the given two-parameter family of functions is the general sol. of the non homogeneous d.E on the indicated interval.

(31) $y'' - 7y' + 10y = 24e^x$; $y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$; $(-\infty, \infty)$

(35) (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, Particular Sol. of $y'' - 6y' + 5y = -9e^{2x}$ and $y'' - 6y' + 5y = 5x^2 + 3x - 16$

(b) use Part (a) to find Particular Sol. of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

and $$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$$

EXERCISE: 3.3

Find the general Solution of given Second order d.E.

(5) $y'' + 8y' + 16y = 0$

(11) $y'' - 4y' + 5y = 0$

(13) $3y'' + 2y' + y = 0$

Ex: 3.3

Solve the given Initial Value Problems.

(29) $y'' + 16y = 0$; $y(0) = 2$; $y'(0) = -2$

(31) $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0$; $y(1) = 0$; $y'(1) = 2$

(35) $y''' + 12y'' + 36y' = 0$; $y(0) = 0$; $y'(0) = 1$; $y''(0) = -7$

SOLUTIONS

EXERCISE : 3.1

3.1.2 Homogeneous Equations

Determine whether the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

(17) $f_1(x) = 5$; $f_2(x) = \cos^2 x$; $f_3(x) = \sin^2 x$

Sol:- Let us assume the given functions are linearly dependent on $(-\infty, \infty)$. Then they should satisfy

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

where c_1, c_2, c_3 are constants, not all zeros.

$$\Rightarrow 5c_1 + c_2 \cos^2 x + c_3 \sin^2 x = 0$$

$$\Rightarrow 5c_1 [\cos^2 x + \sin^2 x] + c_2 \cos^2 x + c_3 \sin^2 x = 0$$

$$\Rightarrow (5c_1 + c_2) \cos^2 x + (5c_1 + c_3) \sin^2 x = 0$$

on comparision with R.H side, we get

$$5c_1 + c_2 = 0 \quad \& \quad 5c_1 + c_3 = 0$$

$$\Rightarrow c_1 = \frac{c_2}{-5} = \frac{c_3}{-5}$$

From proportionality, $c_1 = 1$; $c_2 = -5$; $c_3 = -5$

$$\Rightarrow 5(1) + (-5)\cos^2 x + (-5)\sin^2 x$$

$$\Rightarrow 5 - 5 = 0$$

Hence satisfied.

$\therefore$ Our assumption of linear dependency is true.

Second method for solving the same problem:

Given $f_1(x) = 5$; $f_2(x) = \cos^2 x$; $f_3(x) = \sin^2 x$

$$\Rightarrow \quad D = \begin{vmatrix} 5 & \cos^2 x & \sin^2 x \\ 0 & -\sin 2x & \sin 2x \\ 0 & -2\cos 2x & 2\cos 2x \end{vmatrix}$$

$$\Rightarrow D = 5\left[-2\cos 2x \sin 2x + 2\cos 2x \sin 2x\right] - \cos^2 x(0) + \sin^2 x(0)$$

$$\Rightarrow \quad D = 0$$

$\therefore$ The given functions are linearly dependent.

Ex: 3.1

(19) $f_1(x) = x$; $f_2(x) = x-1$; $f_3(x) = x+3$

Sol: Let us assume the given functions are linearly dependent on $(-\infty, \infty)$. then they should satisfy $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

where c_1, c_2, c_3 are constants, not all zeros.

$$\therefore \quad c_1(x) + c_2(x-1) + c_3(x+3) = 0$$

$$\Rightarrow (c_1 + c_2 + c_3)x + (3c_3 - c_2) = 0$$

On comparision, $\quad c_1 + c_2 + c_3 = 0 \quad -(a)$

and $3c_3 - c_2 = 0 \Rightarrow c_2 = 3c_3$

$$\Rightarrow c_3 = \frac{c_2}{3}$$

From (a), $\quad c_1 + 4c_3 = 0 \Rightarrow c_3 = -\frac{c_1}{4}$

$$\therefore \quad \frac{c_1}{(-4)} = \frac{c_2}{3} = c_3$$

Hence from proportionality, $c_1 = -4$; $c_2 = 3$; $c_3 = 1$

$\Rightarrow \quad (-4)x + 3(x-1) + 1(x+3)$

$= -4x + 3x - 3 + x + 3$

$= 0$

Hence satisfied.

$\therefore$ Our assumption of linear dependency is true.

Second Method:

Given $f_1(x) = x$; $f_2(x) = x-1$; $f_3(x) = x+3$.

$$\Rightarrow \quad D = \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$\Rightarrow D = x(0) - (x-1)(0) + (x+3)(0) = 0$

$\therefore$ The given set of functions are linearly dependent.

31.3 Nonhomogeneous Equations

Verify that the given two-parameter family of functions is the general sol. of the nonhomogeneous d.e on the indicated interval.

(31) $y'' - 7y' + 10y = 24e^x$; $y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$; $(-\infty, \infty)$.

Sol:- Given $y'' - 7y' + 10y = 24e^x$ — (a)

The general sol. of this would be

$y = y_c + y_p$

but it is given that $y = \underbrace{\left[c_1 e^{2x} + c_2 e^{5x}\right]}_{y_c} + \underbrace{6e^x}_{y_p}$

where $y_c \to$ Complimentary Sol. of the homogeneous part

$y_p \to$ particular Sol.

The homogeneous part of (a) is $y'' - 7y' + 10y = 0$

From $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$

let $y_1 = e^{2x}$; $y_2 = e^{5x}$

then
$$W = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{7x} - 2e^{7x} = 3e^{7x} \neq 0$$

and $y_1 = e^{2x}$ & $y_2 = e^{5x}$ also satisfy $y'' - 7y' + 10y = 0$

$\Rightarrow$ y_1 & y_2 form a fundamental set of Solutions of the associated homogeneous eq.

and the assumed y_p i.e $y_p = 6e^{x}$

satisfies $y'' - 7y' + 10y = 24e^{x}$.

$$y_p'' - 7y_p' + 10y_p = 6e^{x} - 42e^{x} + 60e^{x}$$
$$= 24e^{x}.$$

$\therefore$ The assumed $y_p = 6e^{x}$ is the Particular Sol. of the nonhomogeneous eq.

EX: 3.1.3

(35) (a) Verify that $y_{p_1} = 3e^{2x}$ & $y_{p_2} = x^2 + 3x$, are, respectively Particular sol. of $y'' - 6y' + 5y = -9e^{2x}$ and $y'' - 6y' + 5y = 5x^2 + 3x - 16$

Sol:- Given $y_{p_1} = 3e^{2x}$

$$\Rightarrow y'_{p_1} = 6e^{2x} \text{ and } y''_{p_1} = 12e^{2x}$$

So, $y''_p - 6y'_p + 5y_p = 12e^{2x} - 36e^{2x} + 15e^{2x}$

$$= -9e^{2x}$$

and $y_{p_2} = x^2 + 3x$

$$\Rightarrow y'_{p_2} = 2x + 3 \;\&\; y''_{p_2} = 2$$

So, $y''_p - 6y'_p + 5y_p = 2 - 6(2x+3) + 5(x^2+3x)$

$$= 2 - 12x - 18 + 5x^2 + 15x$$

$$= 5x^2 + 3x - 16.$$

$\therefore$ $y_{p_1} = 3e^{2x}$ & $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of $y'' - 6y' + 5y = -9e^{2x}$ and $y'' - 6y' + 5y = 5x^2 + 3x - 16$

35 (b) Use part (a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x} \quad -(i)$$

and $y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x} \quad -(ii)$

Sol:- By superposition principle for nonhomogeneous eq.s, a particular sol. of $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$ is

$$y_p = 3e^{2x} + x^2 + 3x$$

Now, the eq. $y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$

Can be written as, $y'' - 6y' + 5y = -2\left[5x^2 + 3x - 16\right] - \frac{1}{9}\left[-9e^{2x}\right]$

$\therefore$ The R.H part of (ii) is a linear combination of R.H part of (i).

i.e $c_1 g_1(x) + c_2 g_2(x)$

where $g_1(x) = 5x^2+3x-16$

$g_2(x) = -9e^{2x}$

and $c_1 = -2$; $c_2 = -\frac{1}{9}$

$\therefore$ The particular sol. takes the form

$$y_p = c_1 y_{p_2} + c_2 y_{p_2}$$

$$= -2\left[x^2+3x\right] + \left(-\frac{1}{9}\right)\left(3e^{2x}\right)$$

$$\Rightarrow y_p = -2x^2 - 6x - \frac{1}{3}e^{2x}$$

EXERCISE : 3.3

Find the general sol. of the given second order d.E.

(5) $y'' + 8y' + 16y = 0$

Sol: Given $y'' + 8y' + 16y = 0$ — (1)

let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore$ From (1),

$$m^2 e^{mx} + 8me^{mx} + 16e^{mx} = 0$$

$$\Rightarrow e^{mx}\left[m^2 + 8m + 16\right] = 0$$

$e^{mx} \neq 0 \Rightarrow m^2 + 8m + 16 = 0$

$\Rightarrow m = -4, -4 \Rightarrow$ repeated roots.

$\therefore$ The general sol. is $y = c_1 e^{-4x} + c_2 x e^{-4x}$

Ex: 3.3

(11) $y'' - 4y' + 5y = 0$

Sol:- Given $y'' - 4y' + 5y = 0$ — (a)

Let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore$ From (1) : $m^2 e^{mx} - 4me^{mx} + 5e^{mx} = 0$

$\Rightarrow e^{mx}[m^2 - 4m + 5] = 0$

$e^{mx} \neq 0 \Rightarrow m^2 - 4m + 5 = 0$

$\Rightarrow m = 2 \pm i$

$\therefore$ The general sol. is $y = e^{2x}[C_1 \cos x + C_2 \sin x]$

(13) $3y'' + 2y' + y = 0$

Sol:- Given $3y'' + 2y' + y = 0$ — (i)

Let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore$ from (i) : $3m^2 e^{mx} + 2me^{mx} + e^{mx} = 0$

$\Rightarrow e^{mx}[3m^2 + 2m + 1] = 0$

$e^{mx} \neq 0 \Rightarrow 3m^2 + 2m + 1 = 0$

$\Rightarrow m = \frac{-1 \pm \sqrt{2}i}{3} = \frac{-1}{3} \pm \frac{\sqrt{2}}{3} i$

$\therefore$ The general sol. is

$$y = e^{-x/3}\left[C_1 \cos\left(\frac{\sqrt{2}x}{3}\right) + C_2 \sin\left(\frac{\sqrt{2}x}{3}\right)\right]$$

Ex: 3.3

Solve the given Initial value Problems.

(29) $y'' + 16y = 0$; $y(0) = 2$; $y'(0) = -2$

Sol: Given $y'' + 16y = 0$ — (a)

Let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore$ From (a) : $e^{mx}\left[m^2 + 16\right] = 0$

$e^{mx} \neq 0 \Rightarrow m^2 + 16 = 0 \Rightarrow m = \pm 4i$

$\therefore$ The general Sol. is $y = c_1 \cos 4x + c_2 \sin 4x$

Given $y(0) = 2 \Rightarrow$ at $x = 0$, $y = 2$

$\therefore c_1 \cos(0) + c_2 \sin(0) = 2$

$\Rightarrow c_1 = 2$

and $y'(0) = -2 \Rightarrow$ at $x = 0$, $y' = -2$

$y = c_1 \cos 4x + c_2 \sin 4x$

$\Rightarrow y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

$y'(0) = -2$

$\Rightarrow -4c_1 \sin(0) + 4c_2 \cos(0) = -2$

$\Rightarrow 4c_2 = -2$

$\Rightarrow c_2 = -1/2$

$\therefore y = 2\cos 4x - \frac{1}{2}\sin 4x$

(31) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0$; $y(1) = 0$; $y'(1) = 2$

Sol: Given $y'' - 4y' - 5y = 0$ — (a)

Let $y = e^{mt}$ $\Rightarrow y' = me^{mt}$ & $y'' = m^2 e^{mt}$

$\therefore$ from (I) : $e^{mt}[m^2 - 4m - 5] = 0$

$e^{mt} \neq 0 \Rightarrow m^2 - 4m - 5 = 0$

$\Rightarrow m_1 = 5 ; m_2 = -1$

$\therefore$ The general sol. is $y = c_1 e^{5t} + c_2 e^{-t}$

Given $y(1) = 0 \Rightarrow$ at $t = 1$, $y = 0$

$\therefore c_1 e^5 + c_2 e^{-1} = 0$ — (i)

and $y'(1) = 2 \Rightarrow$ at $t = 1$, $y' = 2$

$y = c_1 e^{5t} + c_2 e^{-t}$

$\Rightarrow y' = 5c_1 e^{5t} - c_2 e^{-t}$

$y'(1) = 2$

$\Rightarrow 5c_1 e^5 - c_2 e^{-1} = 2$ — (ii)

Solving (i) & (ii), we get

$c_1 = \frac{e^{-5}}{3}$; $c_2 = \frac{-e}{3}$

$\therefore y = \frac{e^{-5}}{3} e^{5t} - \frac{e}{3}(e^{-t})$

$\Rightarrow y = \frac{1}{3}\left[e^{5(t-1)} - e^{(1-t)}\right]$

Ex: 3.3

(35) $y''' + 12y'' + 36y' = 0$; $y(0) = 0$; $y'(0) = 1$; $y''(0) = -7$

Sol:- Given $y''' + 12y'' + 36y' = 0$ — (a)

Let $y = e^{mx} \Rightarrow y' = me^{mx}$; $y'' = m^2 e^{mx}$ & $y''' = m^3 e^{mx}$

$\therefore$ From (a): $e^{mx}\left[m^3 + 12m^2 + 36m\right] = 0$

$e^{mx} \neq 0 \Rightarrow m\left[m^2 + 12m + 36\right] = 0$

$m = 0$ and $m^2 + 12m + 36 = 0$

$\Rightarrow m = -6, -6$

$\therefore$ The general sol. is $y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$.

Given $y(0) = 0 \Rightarrow$ at $x = 0,\ y = 0$

$\Rightarrow c_1 + c_2 = 0$ — (i)

$y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$

$\Rightarrow y' = -6c_2 e^{-6x} + c_3\left[x(-6e^{-6x}) + e^{-6x}\right]$

$\Rightarrow y' = -6c_2 e^{-6x} + c_3 e^{-6x}\left[-6x + 1\right]$

and $y'' = 36c_2 e^{-6x} + c_3\left[e^{-6x}(-6) + (-6x+1)\left[-6e^{-6x}\right]\right]$

$= 36c_2 e^{-6x} + c_3\left[-6e^{-6x} + 36xe^{-6x} - 6e^{-6x}\right]$

$\Rightarrow y'' = 36c_2 e^{-6x} + c_3\left[-12e^{-6x} + 36xe^{-6x}\right]$

$y'(0) = 1$: $\quad -6c_2 + c_3 = 1$ — (ii)

$y''(0) = -7$: $\quad 36c_2 - 12c_3 = -7$ — (iii)

On solving (i), (ii) & (iii), we get

$c_1 = \frac{5}{36}$; $c_2 = \frac{-5}{36}$; $c_3 = \frac{1}{6}$

$\therefore y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$