MAE3360 HW#4
ENGINEERING ANALYSIS

Exercises 3.5 – Variation of Parameters

3. $y'' + y = \sin x$

$\therefore m^2 + 1 = 0$

$\therefore m = \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

Let $y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ -\sin x & \cos x \end{vmatrix} = \sin^2 x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x$$

$u_1' = -\sin^2 x$

$\therefore u_1' = -\dfrac{(\cos 2x - 1)}{2}$

$\therefore u_1 = \dfrac{\sin 2x}{4} - \dfrac{x}{2}$

$u_2' = \cos x \sin x$

$u_2 = \int \cos x \sin x \, dx$

$= -\frac{1}{2}\cos^2 x$

[All three answers for u_2 are correct]

$$\therefore \boxed{y = c_1\cos x + c_2\sin x + \frac{1}{4}\sin 2x\cdot\cos x - \frac{1}{2}x\cos x - \frac{1}{2}\cos^2 x\sin x}$$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

$m^2 - 9 = 0 \qquad \therefore m = 3 \text{ and } m = -3$

$\therefore y_c = c_1 e^{3x} + c_2 e^{-3x}$

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6$$

$$f(x) = \frac{9x}{e^{3x}}$$

$u_1' = -\dfrac{f(x)y_2}{W}$ $\qquad\qquad$ $u_2' = \dfrac{f(x)y_1}{W}$

$\therefore u_1' = \dfrac{-\frac{9x}{e^{3x}}\cdot e^{-3x}}{-6}$ $\qquad\qquad$ $\therefore u_2' = \dfrac{e^{3x}\cdot\frac{9x}{e^{3x}}}{-6}$

$\therefore u_1' = \dfrac{3}{2}xe^{-6x}$ $\qquad\qquad$ $\therefore u_2' = -\dfrac{3}{2}x$

$\therefore u_1 = -\dfrac{1}{24}e^{-6x} - \dfrac{1}{4}xe^{-6x}$ $\qquad\qquad$ $\therefore u_2 = -\dfrac{3}{4}x^2$

$$\therefore \boxed{y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{24}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2 e^{-3x}}$$

③

19. $4y'' - y = xe^{x/2}$

$4m^2 - 1 = 0 \quad m = \pm 1/2$

$\therefore y_c = c_1 e^{x/2} + c_2 e^{-x/2}$

$$W = \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -1$$

$f(x) = \dfrac{xe^{x/2}}{4}$ note

$u_1' = -\dfrac{f(x)y_2}{W}$ $\qquad$ $u_2' = \dfrac{f(x)y_1}{W}$

$= -\dfrac{\frac{xe^{x/2}}{4} \cdot e^{-x/2}}{-1}$ $\qquad$ $= \dfrac{\frac{xe^{x/2}}{4} \cdot e^{x/2}}{-1}$

$= \dfrac{x}{4}$ $\qquad$ $= \dfrac{-xe^{x}}{4}$

$\therefore u_1 = \dfrac{x^2}{8}$ $\qquad$ $\therefore u_2 = \dfrac{-xe^x}{4} + \dfrac{e^x}{4}$

$$\boxed{\therefore y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}}$$

$y(0) = c_1 + c_2 + \frac{1}{4} = 1 \qquad \therefore c_1 + c_2 = \frac{3}{4}$ — ①

$y' = \frac{1}{2}c_1 e^{x/2} - \frac{1}{2}c_2 e^{-x/2} + \frac{1}{16}x^2 e^{x/2} + \frac{1}{4}x^2 e^{x/2} - \frac{1}{4}e^{x/2}$

$-\frac{1}{4}\left(\frac{1}{2}\right)xe^{x/2} + \frac{1}{8}e^{x/2}$

Using the conditions,

$y(0)=1$

$y'(0)=0$

$C_1 = \frac{1}{2}, \; C_2 = \frac{1}{4}$ $\star$

$$\therefore y = \frac{1}{4}e^{-x/2} + \frac{1}{2}e^{x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$$

$\star$ From $y(0)=1$, we get

$C_1 + C_2 = \frac{3}{4}$ (Refer eqn 1)

From $y'(0)=0 \Rightarrow$

$$y'(0) = \frac{1}{2}C_1 - \frac{1}{2}C_2 - \frac{1}{4} + \frac{1}{8} = 0$$

$$\therefore \frac{1}{2}(C_1 - C_2) = \frac{1}{8}$$

$$\therefore C_1 - C_2 = \frac{1}{4} \quad \text{—— (2)}$$

Solving 1 & 2

$$\begin{aligned} C_1 + C_2 &= 3/4 \\ + \quad C_1 - C_2 &= 1/4 \\ 2C_1 &= 1 \qquad \therefore C_1 = \frac{1}{2} \end{aligned}$$

$$\therefore C_2 = \frac{1}{4}$$

$$\boxed{\therefore y = \frac{1}{2}e^{x/2} + \frac{1}{4}e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}}$$

19. ~~$4y'' - y = x$~~

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

$m^2 + 2m - 8 = 0$

$\therefore m = 2 \;\&\; m = -4$

$\therefore y_c = C_1 e^{2x} + C_2 e^{-4x}$

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}$$

$f(x) = 2e^{-2x} - e^{-x}$

$u_1' = -\dfrac{f(x)\, y_2}{W}$ $\qquad$ $u_2' = \dfrac{f(x)\, y_1}{W}$

$\therefore u_1' = \dfrac{-e^{-4x}(2e^{-2x} - e^{-x})}{-6e^{-2x}}$ $\qquad$ $\therefore u_2' = \dfrac{e^{2x}(2e^{-2x} - e^{-x})}{-6e^{-2x}}$

$\therefore u_1' = \dfrac{1}{3} e^{-4x} - \dfrac{1}{6} e^{-3x}$ $\qquad$ $\therefore u_2' = \dfrac{-1}{3} e^{2x} + \dfrac{1}{6} e^{3x}$

$\therefore u_1 = \dfrac{-1}{12} e^{-4x} + \dfrac{1}{18} e^{-3x}$ $\qquad$ $\therefore u_2 = \dfrac{1}{18} e^{3x} - \dfrac{1}{6} e^{2x}$

$\therefore y = C_1 e^{2x} + C_2 e^{-4x} - \dfrac{1}{12} e^{-2x} + \dfrac{1}{18} e^{-x} + \dfrac{1}{18} e^{-x} - \dfrac{1}{6} e^{-2x}$

$$\boxed{\therefore y = C_1 e^{2x} + C_2 e^{-4x} - \dfrac{1}{4} e^{-2x} + \dfrac{1}{9} e^{-x}}$$

$$\therefore y' = 2c_1 e^{2x} + (-4c_2)e^{-4x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x}$$

$$\therefore y(0) = c_1 + c_2 - \frac{1}{4} + \frac{1}{9} = 1$$

$$\therefore c_1 + c_2 = \frac{41}{36} \quad \text{— } (1)$$

$$y'(0) = 2c_1 - 4c_2 + \frac{1}{2} - \frac{1}{9} = 0$$

$$\therefore 2c_1 - 4c_2 = -\frac{7}{18} \quad \text{— } (2)$$

Solving (1) & (2), we get

$$2c_1 + 2c_2 = \frac{41}{18}$$

$$2c_1 - 4c_2 = -\frac{7}{18}$$

(1) − (2) gives

$$6c_2 = \frac{48}{18} \qquad \therefore \boxed{c_2 = \frac{4}{9}}$$

$$\therefore c_1 = \frac{25}{36}$$

$$\boxed{\therefore y = \frac{25}{36}e^{2x} + \frac{4}{9}e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}}$$