SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 04

DUE ON 02/22/08

QUESTIONS:

Solve the given d.e by Undetermined Coefficients.

(1) $16y'' + 81y = 27$

(2) $y'' + 2y' + 2y = 4x^2$

(3) $y'' + 2y' + y = x + e^{-x}$

(4) $y'' - y' - 2y = 40\sin^2 x$

(5) $y'' + 2y' = 49e^x \sin 2x$

(6) Solve the given Initial Value problems

$$y'' - y = 9xe^{2x} \; ; \; y(0) = 0 \; ; \; y'(0) = 7$$

Solve the given differential equation,

(7) $x^2y'' + 3xy' + y = 0$

(8) $x^2y'' + xy' + 16y = 0$

(9) $x^2y'' - x(2m-1)y' + (m^2 + K^2)y = 0$

(10) $x^2y'' + xy' - m^2y = 0$

where m & K are Positive Constants in each Case.

SOLUTIONS

Solve the given d.E by Undetermined Coefficients

(1) $16y'' + 81y = 27$

Sol:- From homogeneous part, $16y'' + 81y = 0$

$$\text{Let } y = e^{mx} \Rightarrow y' = me^{mx} \ \& \ y'' = m^2 e^{mx}$$

$$\therefore \ e^{mx}[16m^2 + 81] = 0$$

$$e^{mx} \neq 0 \Rightarrow 16m^2 + 81 = 0 \Rightarrow m = \pm \frac{9}{4} i$$

$$\therefore \ y_c = c_1 \cos\left(\frac{9x}{4}\right) + c_2 \sin\left(\frac{9x}{4}\right)$$

On Right-hand side, we have '27' which is a Constant.

$\therefore$ let particular Sol. be

$$y_p = A$$

$$\Rightarrow y_p' = 0 \ \& \ y_p'' = 0$$

Substituting in $16y_p'' + 81y_p = 27$

$$\Rightarrow 81A = 27$$

$$\Rightarrow A = 1/3$$

$$\therefore y_p = 1/3$$

$\therefore$ The general Sol. is $y = y_c + y_p$

$$\Rightarrow y = c_1 \cos\left(\frac{9x}{4}\right) + c_2 \sin\left(\frac{9x}{4}\right) + \frac{1}{3}$$

(2) $y'' + 2y' + 2y = 4x^2$

Sol:- From homogeneous part, $y'' + 2y' + 2y = 0$

let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore \quad e^{mx}[m^2 + 2m + 2] = 0$

$e^{mx} \neq 0 \Rightarrow m^2 + 2m + 2 = 0$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$\therefore \quad y_c = e^{-x}[C_1 \cos x + C_2 \sin x]$

Let the particular Sol. be

$y_p = Ax^2 + Bx + C$

$\Rightarrow y_p' = 2Ax + B$

$\Rightarrow y_p'' = 2A$

Substituting y_p into original DE yields $y_p'' + 2y_p' + 2y_p = 4x^2$

$\Rightarrow 2A + 2(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$

$\Rightarrow 2A + 4Ax + 2B + 2Ax^2 + 2Bx + 2C = 4x^2$

On Comparision,

x^2: $2A = 4 \Rightarrow A = 2$

x: $4A + 2B = 0 \Rightarrow 8 + 2B = 0 \Rightarrow B = -4$

Const: $2(A + B + C) = 0 \Rightarrow 2(2 - 4 + C) = 0$

$\Rightarrow C = 2$

$\therefore \quad y_p = 2x^2 - 4x + 2$

$\therefore$ The general Sol. is $y = y_c + y_p$

$\therefore \quad y = e^{-x}[C_1 \cos x + C_2 \sin x] + 2x^2 - 4x + 2$

(3) $y'' + 2y' + y = x + e^{-x}$

Sol: From homogeneous Part, $y'' + 2y' + y = 0$

let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$\therefore e^{mx}[m^2 + 2m + 1] = 0$

$e^{mx} \neq 0 \Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

$\therefore y_c = c_1 e^{-x} + c_2 x e^{-x}$

Let particular sol. be $y_p = Ax + B + cx^2 e^{-x}$

Note: If we Consider $y_p = Ax + B + ce^{-x}$, the term ce^{-x} is duplicated in y_c. Hence the above form has to taken into account.

[Refer to P.No 129 in Text book under "A Glitch in the method" also Example: 8 on Page 131]

$\therefore y_p' = A + c[-x^2 e^{-x} + 2xe^{-x}]$

$\Rightarrow y_p'' = c[(x^2 - 4x + 2)e^{-x}]$

Substituting y_p into original D.E yields

$y_p'' + 2y_p' + y_p = x + e^{-x}$

$\Rightarrow ce^{-x}[x^2 - 4x + 2] + 2[A + c(-x^2 e^{-x} + 2xe^{-x})] + Ax + B + cx^2 e^{-x} = x + e^{-x}$

on solving, $Ax + B + 2A + 2ce^{-x} = x + e^{-x}$

x: $A = 1$

e^{-x}: $2c = 1 \Rightarrow c = 1/2$

Const: $2A + B = 0 \Rightarrow 2 + B = 0 \Rightarrow B = -2$

$$\therefore y_p = x - 2 + \frac{x^2 e^{-x}}{2}$$

$\therefore$ The general sol. is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + (x-2) + \frac{1}{2} x^2 e^{-x}$$

(4) $y'' - y' - 2y = 40 \sin^2 x$

Sol:- Since $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$\therefore$ The given d.e can be written as

$$y'' - y' - 2y = 40\left[\frac{1 - \cos 2x}{2}\right]$$

$$\Rightarrow y'' - y' - 2y = 20 - 20\cos 2x$$

Considering the homogeneous part,

$$y'' - y' - 2y = 0$$

Let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$$\therefore e^{mx}\left[m^2 - m - 2\right] = 0$$

$$e^{mx} \neq 0 \Rightarrow m^2 - m - 2 = 0$$

$$\Rightarrow (m+1)(m-2) = 0$$

$$\Rightarrow m = -1, 2$$

$$\therefore y_c = c_1 e^{-x} + c_2 e^{2x}$$

Let the Particular Sol. be

$$y_p = A\cos 2x + B\sin 2x + c$$

$$\Rightarrow y_p' = -2A\sin 2x + 2B\cos 2x$$

$$\Rightarrow y_p'' = -4A\cos 2x - 4B\sin 2x$$

Substituting y_p into original D.E yields,

$$y_p'' - y_p' - 2y_p = 20 - 20\cos 2x$$

$$\Rightarrow (-4A\cos 2x - 4B\sin 2x) + 2A\sin 2x - 2B\cos 2x - 2A\cos 2x - 2B\sin 2x - 2c = 20 - 20\cos 2x$$

$$\Rightarrow (-4A - 2B - 2A)\cos 2x + (-4B + 2A - 2B)\sin 2x - 2c = 20 - 20\cos 2x$$

$\cos 2x$: $\quad -6A - 2B = -20$

$\sin 2x$: $\quad -6B + 2A = 0$

on Solving, $A = 3$; $B = 1$

Const: $\quad -2c = 20 \Rightarrow c = -10$.

$$\therefore y_p = 3\cos 2x + \sin 2x - 10$$

$\therefore$ The general Sol. is $\quad y = y_c + y_p$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{2x} + 3\cos 2x + \sin 2x - 10.$$

(5) $y'' + 2y' = 49e^{x}\sin 2x$

Sol: From homogeneous Part, $y'' + 2y' = 0$

Let $y = e^{mx} \Rightarrow y' = me^{mx}$ & $y'' = m^2 e^{mx}$

$$\therefore e^{mx}\left[m^2 + 2m\right] = 0$$

$e^{mx} \neq 0 \quad \Rightarrow m(m+2) = 0$

$\Rightarrow m = 0, -2$

$\therefore y_c = C_1 + C_2 e^{-2x}$

Let particular sol. be $y_p = e^x (A\cos 2x + B\sin 2x)$

$\Rightarrow y_p' = A\left[e^x(\cos 2x - 2\sin 2x)\right] + B\left[e^x(2\cos 2x + \sin 2x)\right]$

$\Rightarrow y_p'' = A\left[-e^x(3\cos 2x + 4\sin 2x)\right] + B\left[e^x(4\cos 2x - 3\sin 2x)\right]$

From $y_p'' + 2y_p' = 49 e^x \sin 2x$

$\Rightarrow \left[-3Ae^x \cos 2x - 4Ae^x \sin 2x + 4Be^x \cos 2x - 3Be^x \sin 2x\right]$

$+ 2Ae^x \cos 2x - 4Ae^x \sin 2x + 4Be^x \cos 2x + 2Be^x \sin 2x$

$= 49 e^x \sin 2x$

On Comparision:

$e^x \cos 2x$: $-3A + 4B + 2A + 4B = 0$

$\Rightarrow -A + 8B = 0 \quad -(a)$

$e^x \sin 2x$: $-4A - 3B - 4A + 2B = 49$

$\Rightarrow -8A - B = 49 \quad -(b)$

Solving (a) & (b)

$A = \frac{-392}{65}$; $B = \frac{-49}{65}$

$\therefore y_p = e^x \left[\left(-\frac{392}{65}\right)\cos 2x + \left(\frac{-49}{65}\right)\sin 2x\right]$

$\therefore$ The general Sol. is $y = y_c + y_p = C_1 + C_2 e^{-2x} + e^x \left[\left(\frac{-392}{65}\right)\cos 2x + \left(\frac{-49}{65}\right)\sin 2x\right]$

(6) Solve the given initial value problem.

$$y'' - y = 9xe^{2x} \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = 7.$$

Sol: Given $y'' - y = 9xe^{2x}$

Considering the homogeneous part, $y'' - y = 0$

$$\text{let } y = e^{mx}$$

$$\Rightarrow y' = me^{mx} \;\&\; y'' = m^2 e^{mx}$$

$$\therefore e^{mx}[m^2 - 1] = 0$$

$$e^{mx} \neq 0 \Rightarrow m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

$$\therefore y_c = c_1 e^{x} + c_2 e^{-x}$$

let particular sol. be, $y_p = (Ax + B)e^{2x}$

$$\Rightarrow y_p' = (Ax+B)\,2e^{2x} + e^{2x}(A)$$

$$\Rightarrow y_p'' = 2\left[(Ax+B)\,2e^{2x} + e^{2x}(A)\right] + 2Ae^{2x}$$

On substitution into $y_p'' - y_p = 9xe^{2x}$

$$4e^{2x}(Ax+B) + 2Ae^{2x} + 2Ae^{2x} - (Ax+B)e^{2x} = 9xe^{2x}$$

$$\Rightarrow 4Axe^{2x} + 4Be^{2x} + 4Ae^{2x} - Axe^{2x} - Be^{2x} = 9xe^{2x}$$

$$\Rightarrow 3Axe^{2x} + (4A + 3B)e^{2x} = 9xe^{2x}$$

On Comparision:

$$\underset{\sim}{xe^{2x}} : \; 3A = 9 \Rightarrow A = 3$$

e^{2x} : $4A + 3B = 0$

$\Rightarrow 12 + 3B = 0 \Rightarrow B = -4$

$\therefore y_p = (3x-4)e^{2x}$

$\therefore$ The general sol. is $y = y_c + y_p$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + (3x-4)e^{2x}$

Given $y(0) = 0 \Rightarrow$ at $x=0, y=0$

$\Rightarrow c_1 + c_2 - 4 = 0 \Rightarrow c_1 + c_2 = 4$ — (a)

$y'(0) = 7 \Rightarrow$ at $x = 0, y' = 7$

$\therefore y' = c_1 e^{x} - c_2 e^{-x} + 2(3x-4)e^{2x} + e^{2x}(3)$

$y'(0) \equiv c_1 - c_2 - 8 + 3 = 7$

$\Rightarrow y'(0) \equiv c_1 - c_2 = 12$ — (b)

Solving (a) & (b)

$$\begin{array}{l} c_1 + c_2 = 4 \\ c_1 - c_2 = 12 \\ \hline \end{array}$$

$\Rightarrow c_1 = 8$; $c_2 = -4$

$\therefore y = 8e^{x} - 4e^{-x} + (3x-4)e^{2x}$

→ Solve the given differential equation:

(7) $x^2y'' + 3xy' + y = 0$

Sol: let $y = x^m \Rightarrow y' = mx^{m-1}$ & $y'' = m(m-1)x^{m-2}$

$\therefore$ From the given eq.,

$$x^2\left[m(m-1)x^{m-2}\right] + 3x\left[mx^{m-1}\right] + x^m = 0$$

$$\Rightarrow x^m\left[m(m-1) + 3m + 1\right] = 0$$

$$x^m \neq 0 \Rightarrow m^2 + 2m + 1 = 0$$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1$$

$\therefore$ Repeated roots

$\Rightarrow$ The general Sol. is

$$y = c_1 x^{-1} + c_2 x^{-1} \ln x$$

(8) $x^2 y'' + xy' + 16y = 0$

Sol:- Let $y = x^m \Rightarrow y' = mx^{m-1}$ & $y'' = m(m-1)x^{m-2}$

$\therefore$ From the given eq.,

$$x^2\left[m(m-1)x^{m-2}\right] + x\left[mx^{m-1}\right] + 16x^m = 0$$

$$\Rightarrow x^m\left[m(m-1) + m + 16\right] = 0$$

$$\Rightarrow x^m \neq 0 \Rightarrow m^2 + 16 = 0$$

$$\Rightarrow m = \pm 4i$$

$\therefore$ The general Sol. is $y = c_1 x^{4i} + c_2 x^{-4i}$

(or) $y = c_1 \cos(4\ln x) + c_2 \sin(4\ln x)$

(9) $x^2y'' - x(2m-1)y' + (m^2+k^2)y = 0$ where m & k are positive constants.

Sol: Let $y = x^r \Rightarrow y' = rx^{r-1}$; $y'' = r(r-1)x^{r-2}$

$\therefore$ From the given eq.,

$$x^2\left[r(r-1)x^{r-2}\right] - x(2m-1)\left[rx^{r-1}\right] + (m^2+k^2)x^r = 0$$

$$\Rightarrow x^r\left[r(r-1) - r(2m-1) + m^2 + k^2\right] = 0$$

$$x^r \neq 0 \Rightarrow r^2 - \cancel{r} - 2mr + \cancel{r} + m^2 + k^2 = 0$$

$$\Rightarrow r^2 - 2mr + (m^2+k^2) = 0$$

$$\Rightarrow r = m \pm ki$$

$\therefore$ The general sol. is $y = x^m\left[c_1\cos(k\ln x) + c_2\sin(k\ln x)\right]$

(10) $x^2y'' + xy' - m^2y = 0$ where m is a positive constant.

Sol: Let $y = x^r \Rightarrow y' = rx^{r-1}$ & $y'' = r(r-1)x^{r-2}$

$\therefore$ From the given eq.,

$$x^2\left[r(r-1)x^{r-2}\right] + x\left[rx^{r-1}\right] - m^2x^r = 0$$

$$\Rightarrow x^m\left[r(r-1) + r - m^2\right] = 0$$

$$x^m \neq 0 \Rightarrow r^2 - m^2 = 0 \Rightarrow r = \pm m$$

$\therefore$ The general sol. is

$$y = c_1x^m + c_2x^{-m}$$