SUB : MAE 3360

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SOLUTIONS TO (SUPPLEMENTAL) ASSIGNMENT # 05

DUE ON 03/12/'08

QUESTIONS

Solve the given d.e by variation of parameter.

(1) $x^2y'' + 4xy' + 2y = 4\ln x$

(2) $x^2y'' + 6xy' + 6y = 4e^{2x}$

(3) $x^2y'' + 4xy' + 2y = \cos x$

(4) $y'' - 2y' + y = \dfrac{e^x}{x}$; $y(1) = 0$; $y'(1) = 1$

(5) $y'' + 4y = \sin^2 2x$; $y(\pi) = 0$; $y'(\pi) = 0$

SOLUTIONS

Solve the given d.E by variation of Parameters.

(1) $x^2 y'' + 4xy' + 2y = 4\ln x$

Sol:- Given $x^2 y'' + 4xy' + 2y = 4\ln x \quad -(1)$

The auxiliary eq. with $y = x^m$ for left hand side

$\Rightarrow x^m [m^2 + 3m + 2] = 0$

$x^m \neq 0 \Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1$

$\therefore y_c = c_1 x^{-2} + c_2 x^{-1}$

$\Rightarrow y_1 = x^{-2} ; \; y_2 = x^{-1}.$

From (1), dividing both sides by x^2,

$$y'' + \frac{4y'}{x} + \frac{2y}{x^2} = \frac{4\ln x}{x^2} \quad -(1a)$$

$$w(x^{-2}, x^{-1}) = \begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix} = x^{-4}.$$

$\Rightarrow W = x^{-4}.$

with $f(x) = \dfrac{4\ln x}{x^2}$,

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{4\ln x}{x^2} & -x^{-2} \end{vmatrix} = -\frac{4\ln x}{x^3}$$

$$W_2 = \begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & \frac{4\ln x}{x^2} \end{vmatrix} = \frac{4\ln x}{x^4}$$

From $u'y_1 + v'y_2 = 0$

and $u'y_1' + v'y_2' = f(x)$,

we have $u' = \frac{W_1}{W} = \frac{(-4\ln x)/x^3}{x^{-4}} = -4x\ln x$

$\Rightarrow u(x) = -4\int x\ln x\,dx = -4\int \ln x \cdot x\,dx = -2x^2\left(\ln x - \frac{1}{2}\right)$

and $v' = \frac{W_2}{W} = \frac{4\ln x/x^4}{x^{-4}} = 4\ln x$

$\Rightarrow v(x) = 4\int \ln x\,dx = 4x(\ln x - 1)$

$\therefore y_p = -2\cancel{x^2}\left(\ln x - \frac{1}{2}\right)\cancel{x^{-2}} + \cancel{x^{-1}}\left(4\cancel{x}(\ln x - 1)\right)$

$\Rightarrow y_p = -2\ln x + 1 + 4\ln x - 4$

$\Rightarrow y_p = 2\ln x - 3$

The general Sol.,

$y = y_c + y_p$

$\Rightarrow y = c_1 x^{-2} + c_2 x^{-1} + 2\ln x - 3.$

(2) $x^2 y'' + 6xy' + 6y = 4e^{2x}$

Sol:- Given $x^2 y'' + 6xy' + 6y = 4e^{2x}$ — (1)

For the homogeneous part in (1),

let $y = x^m$

$\Rightarrow x^m [m^2 + 5m + 6] = 0$

$x^m \neq 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3.$

$\therefore y_c(x) = c_1 x^{-2} + c_2 x^{-3}.$

let $y_1(x) = x^{-2}$; $y_2(x) = x^{-3}$.

From (1), dividing both sides by x^2,

$$y'' + \frac{6}{x} y' + \frac{6}{x^2} y = \frac{4}{x^2} e^{2x} \quad - (1(a))$$

$$\Rightarrow f(x) = \frac{4}{x^2} e^{2x}$$

$$W(x^{-2}, x^{-3}) = \begin{vmatrix} x^{-2} & x^{-3} \\ -2x^{-3} & -3x^{-4} \end{vmatrix} = -x^{-6}$$

$$W_1 = \begin{vmatrix} 0 & x^{-3} \\ \frac{4e^{2x}}{x^2} & -3x^{-4} \end{vmatrix} = \frac{-4e^{2x}}{x^5}.$$

$$W_2 = \begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & \frac{4e^{2x}}{x^2} \end{vmatrix} = \frac{4e^{2x}}{x^4}$$

From $u' y_1 + v' y_2 = 0$

and $u' y_1' + v' y_2' = f(x)$

we have $u' = \frac{W_1}{W} = \frac{+4e^{2x}/x^5}{+x^{-6}} = 4xe^{2x}$

$$\Rightarrow u(x) = 4\int xe^{2x}\,dx = 4\left[\frac{e^{2x}}{2}\left(x-\frac{1}{2}\right)\right] = 2e^{2x}\left(x-\frac{1}{2}\right)$$

$$v' = \frac{W_2}{W} = \frac{4e^{2x}/x^4}{-x^{-6}} = -4x^2e^{2x}$$

$$\Rightarrow v(x) = -4\int x^2e^{2x}\,dx = -4\left[\frac{e^{2x}}{2}\left(x^2-x+\frac{1}{2}\right)\right]$$

$$\Rightarrow v(x) = -2e^{2x}\left(x^2-x+\frac{1}{2}\right)$$

$$\therefore y_p = 2e^{2x}\left(x-\frac{1}{2}\right)(x^{-2}) + (x^{-3})\left[-2e^{2x}\left(x^2-x+\frac{1}{2}\right)\right]$$

$$= 2e^{2x}\left[\cancel{x^{-1}} - \frac{1}{2x^2} - \cancel{x^{-1}} + x^{-2} - \frac{1}{2x^3}\right]$$

$$= 2e^{2x}\left[\frac{x^{-2}}{2} - \frac{x^{-3}}{2}\right]$$

$$\Rightarrow y_p = \frac{e^{2x}(x-1)}{x^3}$$

$\therefore$ The general Sol;

$$y_g = y_c + y_p = c_1x^{-2} + c_2x^{-3} + \frac{e^{2x}(x-1)}{x^3}.$$

(3) $x^2y'' + 4xy' + 2y = \cos x.$

Sol:- Given $x^2y'' + 4xy' + 2y = \cos x$ — (1)

For the homogeneous part of (1), let $y = x^m$

$$\Rightarrow x^m\left[m^2+3m+2\right] = 0$$

$$x^m \neq 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$$

$\therefore y_c = c_1 x^{-1} + c_2 x^{-2}$

From ①, dividing both sides by x^2,

$$y'' + \frac{4y'}{x} + \frac{2y}{x^2} = \frac{\cos x}{x^2}$$

where $f(x) = \dfrac{\cos x}{x^2}$.

$$W(x^{-1}, x^{-2}) = \begin{vmatrix} x^{-1} & x^{-2} \\ -x^{-2} & -2x^{-3} \end{vmatrix} = -x^{-4}$$

$$\Rightarrow W = -x^{-4}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-2} \\ \frac{\cos x}{x^2} & -2x^{-3} \end{vmatrix} = -\frac{\cos x}{x^4}$$

$$W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{\cos x}{x^2} \end{vmatrix} = \frac{\cos x}{x^3}$$

$$u' = \frac{W_1}{W} = \frac{+\cos x / x^4}{+x^{-4}} = \cos x$$

$$\Rightarrow u(x) = \int \cos x \, dx = \sin x$$

$$v' = \frac{W_2}{W} = \frac{\cos x / x^3}{-x^{-4}} = -x\cos x$$

$$\Rightarrow v(x) = -\int x \cos x \, dx = -(x \sin x + \cos x)$$

$$\therefore y_p = (x^{-1}) \sin x - x^{-2}(x \sin x + \cos x) = -\frac{\cos x}{x^2}$$

$$\therefore y = y_c + y_p = c_1 x^{-1} + c_2 x^{-2} - \frac{\cos x}{x^2}.$$

(4) $y'' - 2y' + y = \dfrac{e^x}{x}$; $y(1) = 0$; $y'(1) = 1$

Sol:- Given $y'' - 2y' + y = \dfrac{e^x}{x}$ — (1)

with $y = e^{mx}$,

the auxiliary eq. is $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0 \Rightarrow m = +1, +1.$$

$$\therefore y_h = c_1 e^{x} + c_2 x e^{x}$$

with $y_1 = e^x$; $y_2 = xe^x$ and $f(x) = \dfrac{e^x}{x}$

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(x+1) \end{vmatrix} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x(x+1) \end{vmatrix} = -e^{2x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x}$$

$$u' = \frac{W_1}{W} = \frac{-e^{2x}}{e^{2x}} = -1$$

$$\Rightarrow u(x) = -x$$

$$v' = \frac{W_2}{W} = \frac{e^{2x}/x}{e^{2x}} = 1/x$$

$$\Rightarrow v(x) = \ln x$$

$\therefore y_p = -xe^x + xe^x \ln x$

$\therefore$ The general sol. is

$y = y_c + y_p = c_1 e^x + c_3 x e^x + x e^x \ln x$ where $c_3 = (c_2 - 1)$

$\Rightarrow y' = c_1 e^x + c_3 e^x (x+1) + e^x[1 + x\ln x + \ln x]$

$\frac{d}{dx}[\underbrace{xe^x}_{u} \underbrace{\ln x}_{v}]$

$= xe^x\left(\frac{1}{x}\right) + \ln x\,[xe^x + e^x]$

$= e^x\,[1 + x\ln x + \ln x]$

Given $y(1) = 0 \Rightarrow$ at $x = 1, y = 0$

$\Rightarrow c_1 e + c_3 e = 0$

$\Rightarrow \boxed{c_1 = -c_3}$ — (a)

and $y'(1) = 1$

$\Rightarrow$ at $x = 1, y' = 1$

$\Rightarrow c_1 e + 2c_3 e + e = 1$

From (a),

$-c_3 e + 2c_3 e = 1 - e \Rightarrow c_3 = \frac{1-e}{e}$

$\Rightarrow c_1 = -c_3 = \frac{e-1}{e}$

$\therefore y = \left(\frac{e-1}{e}\right)e^x + \left(\frac{1-e}{e}\right)e^x(x) + xe^x \ln x$

$\Rightarrow y = e^{x-1}(e-1)(1-x) + xe^x \ln x$

(5) $y'' + 4y = \sin^2 2x$; $y(\pi) = 0$; $y'(\pi) = 0$

sol:- Given $y'' + 4y = \sin^2 2x$ — (1)

For the homogeneous part let $y = e^{mx}$

$$\Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x.$$

and $f(x) = \sin^2 2x$

$$y_1 = \cos 2x \;;\; y_2 = \sin 2x$$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$\Rightarrow W = 2$$

$$W_1 = \begin{vmatrix} 0 & \sin 2x \\ \sin^2 2x & 2\cos 2x \end{vmatrix} = -\sin^3 2x$$

$$W_2 = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sin^2 2x \end{vmatrix} = \sin^2 2x \cos 2x$$

$$u' = \frac{W_1}{W} = \frac{-\sin^3 2x}{2}$$

$$\Rightarrow u(x) = -\frac{1}{2}\int \sin^3 2x \, dx = \frac{-1}{2}\left[\frac{-9\cos 2x + \cos 6x}{24}\right]$$

$$= \frac{-1}{2}\left[\frac{-9\cos 2x + 4\cos^3 2x - 3\cos 2x}{24}\right]$$

$$\Rightarrow u(x) = \frac{1}{4}\cos 2x - \frac{1}{12}\cos^3 2x$$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\cos 6x = \cos 3(2x)$

$\cos 3x = 4\cos^3 x - 3\cos x$

$$v' = \frac{W_2}{W} = \frac{1}{2}\sin^2 2x \cos 2x$$

$$\Rightarrow v(x) = \frac{1}{2}\int \sin^2 2x \cos 2x \, dx$$

let $\sin 2x = t \Rightarrow 2\cos 2x \, dx = dt$

$\Rightarrow \cos 2x \, dx = dt/2$

$$\Rightarrow v(x) = \frac{1}{2}\int t^2 \frac{dt}{2} = \frac{1}{4}\left(\frac{t^3}{3}\right) = \frac{1}{12} t^3$$

$$\Rightarrow v(x) = \frac{1}{12}\sin^3 2x$$

$$\therefore y_p = \left(\frac{1}{4}\cos 2x - \frac{1}{12}\cos^3 2x\right)\cos 2x + \frac{1}{12}\sin^3 2x (\sin 2x)$$

$$\Rightarrow y_p = \frac{1}{4}\cos^2 2x - \frac{1}{12}\cos^4 2x + \frac{1}{12}\sin^4 2x$$

$$= \frac{1}{4}\cos^2 2x - \frac{1}{12}\left[\cos^4 2x - \sin^4 2x\right]$$

$$= \frac{1}{4}\cos^2 2x - \frac{1}{12}\left[(\cos^2 2x - \sin^2 2x)(\cos^2 2x + \sin^2 2x)\right]$$

$$\Rightarrow y_p = \frac{1}{6}\cos^2 2x + \frac{1}{12}\sin^2 2x$$

$\therefore$ The general sol,

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{6}\cos^2 2x + \frac{1}{12}\sin^2 2x.$$

$$\Rightarrow y' = -2c_1 \sin 2x + 2c_2 \cos 2x - \frac{1}{3}\cos 2x \sin 2x$$

$\frac{d}{dx}(\cos^2 2x) = -\frac{2}{3}\cos 2x \sin 2x$

$\frac{d}{dx}(\sin^2 2x) = 4\sin 2x \cos 2x$

Given $y(\pi) = 0 \Rightarrow$ at $x = \pi$, $y = 0$

$$\Rightarrow c_1 + \frac{1}{6} = 0 \Rightarrow c_1 = -\frac{1}{6}$$

$y'(\pi) = 0 \Rightarrow$ at $x = \pi$, $y' = 0$

$$\Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$$

$$\boxed{\therefore y = -\frac{1}{6}\cos 2x + \frac{1}{6}\cos^2 2x + \frac{1}{12}\sin^2 2x}$$