MAE 3360

HW #7 ①

Exercises 6.1 – Euler Methods

6. $y' = x + y^2$, $y(0)=0$, $y(0.5)$? $h = 0.1$

$x_0 = 0$, $y_0 = 0$

$f(x_n, y_n) = x_n + y_n^2$

$y_1^* = y_0 + 0.1(x_0 + y_0^2) = 0$

$y_1 = y_0 + \frac{0.1}{2}(x_0 + y_0^2 + x_1 + y_1^{*2}) = 0 + \frac{0.1}{2}(0 + 0^2 + 0.1 + 0^2) = 0.005$

$y_2^* = 0.015002$

$y_2 = 0.020013$

$y_3^* = 0.04005$

$y_3 = 0.04511$

$y_4^* = 0.07531$

$y_4 = 0.0849$

$y_5^* = 0.12114$

$y_5 = 0.1266$

8. $y' = xy + \sqrt{y}$ $y(0)=1$, $y(0.5)$? $h = 0.1$

$x_0 = 0$, $y_0 = 1$

$f(x_n, y_n) = x_n y_n + \sqrt{y_n}$

$y_1^* = y_0 + 0.1(x_0 y_0 + \sqrt{y_0}) = 1 + 0.1(0.1 + \sqrt{1}) = 1.1$

$\therefore y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore y_1 = 0 + \frac{0.1}{6}(0.0 + 2 \times 0.05 + 2 \times 0.05006 + 0.10002)$

$= 0.005$

Similarly,

for $x_n = 0.1$,

$K_1 = 0.1002$

$K_2 = 0.1501$

$K_3 = 0.15015$

$K_4 = 0.2004$

$y_n = 0.0005$

for $x_n = 0.2$

$K_1 = 0.2004$

$K_2 = 0.2509$

$K_3 = 0.2510$

$K_4 = 0.3020$

$y_n = 0.02002$

for $x_n = \cancel{0.3}\ 0.3$

$K_1 = 0.3020$

$K_2 = 0.3536$

$K_3 = 0.3539$

$K_4 = 0.4065$

$y_n = 0.0451$

for $x_n = 0.4$

$K_1 = 0.4064$

$K_2 = 0.4601$

$K_3 = 0.4607$

$K_4 = 0.51602$

$y_n = 0.0805$

for $x_n = 0.5$

$y_n = 0.12659$

$\therefore y_1 = y_0 + \frac{0.1}{2}(x_0 y_0 + \sqrt{y_0} + x_1 y_1^* + \sqrt{y_1^*}) = 1 + \frac{0.1}{2}(0.1 + \sqrt{1} + 0.1 \times 1.1 + \sqrt{1.1}) = 1.1080$

Similarly,

$y_2^* = 1.2243$

$y_2 = 1.2338\!\!\!/7$

$y_3^* = 1.3694$

$y_3 = 1.3806$

$y_4^* = 1.5395$

$y_4 = 1.5529$

$y_5^* = 1.7397$

$y_5 = 1.7557$

Exercises 6.2 – Runge Kutta Methods.

8. $y' = x + y^2, \quad y(0) = 0, \quad y(0.5)? \quad h = 0.1$

For R K method,

$K_1 = f(x_n, y_n) = x_0 + y_0^2 = 0 + 0 = 0$

$K_2 = f(x_n + \frac{1}{2}h, \; y_n + \frac{1}{2}hK_1) = 0 + \frac{1}{2}0.1 + (0 + \frac{1}{2}0.1 \times 0)^2 = 0.05$

$K_3 = f(x_n + \frac{1}{2}h, \; y_n + \frac{1}{2}hK_2) = 0 + \frac{1}{2}0.1 + (0 + \frac{1}{2}0.1 \times 0.05)^2 = 0.05006$

$K_4 = f(x_n + h, \; y_n + hK_3) = 0 + 0.1 + (0 + 0.1 \times 0.05006)^2$

$= 0.10002$

10. $y' = xy + \sqrt{y}$, $y(0)=1$, $y(0.5)$? $h=0.1$

$k_1 = x_0 y_0 + \sqrt{y_0} = 0 \times 1 + \sqrt{1} = 1$

$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) = \left(x_0 + \frac{0.1}{2}\right)\left(y_0 + \frac{0.1}{2} \times 1\right) + \sqrt{\left(1 + \frac{0.1}{2} \times 1\right)} = 1.0772$

$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) = \left(0 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2} \times 1.0772\right) + \left(1 + \frac{0.1 \times 1.0772}{2}\right)^{1/2}$

$= 1.0792$

$k_4 = f(x_n + h, y_n + hk_3) = (0 + 0.1)(1 + 0.1 \times 1.0793) + (1 + 0.1 \times 1.0792)^{1/2}$
$= 1.1633$

$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{0.1}{6}(1 + 2 \times 1.0772 + 2 \times 1.0792 + 1.1633)$

$= 1.10794$

similarly,

x_n	y_n
→ 0.1	1.1079
0.2	1.2337
0.3	1.3807
0.4	1.5531
0.5	1.7561

Exercises 6.4 — Higher Order Equations & Systems.

2. $x^2y'' - 2xy' + 2y = 0$

$y(1) = 4 \quad y'(1) = u_0 = 9 \quad y(1.2)? \quad h = 0.1$

Let $y' = u$

$\therefore x^2 u' - 2xu + 2y = 0$

$\therefore u' = \dfrac{2u}{x} - \dfrac{2y}{x^2}$

$y_{n+1} = y_n + 0.1 u_n \quad \therefore y_1 = y_0 + 0.1 u_0 = 4 + 0.1 \times 9 = 4.9$

$u_{n+1} = u_n + 0.1 u_n' \quad \therefore u_1 = u_0 + 0.1 (u_n') = 9 + 0.1 [2(9) - 2(4)]$
$= 10$

$y_2 = y_1 + 0.1 u_1 = 4.9 + 0.1 (10) = 5.9$

Now, put $y = x^m$

$\therefore x^2 \; m^2 - 3m + 2 = 0$

$\therefore m = 1 \;\&\; m = 2$

$\therefore y = C_1 x + C_2 x^2 \qquad y' = C_1 + 2C_2 x$

$y(1) = C_1 + C_2 = 4 \qquad y'(1) = C_1 + 2C_2 = 9$

Solving, $C_1 = -1, \; C_2 = 5$

$\therefore y = -x + 5x^2$

at $x = 1.2$, $\quad y = -1.2 + 5(1.2)^2 = 6.0$

4. using the RK4 method,

substitute $y' = u$

$\therefore u' = \frac{2}{x}u - \frac{2}{x^2}y$

Using, $y_{n+1} = y_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$$u_{n+1} = u_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $m_1 = u_n$

$$m_2 = u_2 + \frac{1}{2}hk_1$$

$$m_3 = u_n + \frac{1}{2}hk_2$$

$$m_4 = u_n + \frac{1}{2}hk_3$$

where $k_1 = f(x_n, y_n, u_n)$

$$k_2 = f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hm_1;\ u_n + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}m_2h;\ u_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h,\ y_n + hm_3,\ u_n + hk_3)$$

We obtain the following table,

x_n	$h=0.2$ y_n	$h=0.2$ u_n	$h=0.1$ y_n	$h=0.1$ u_n
1.0	4.0000	9.0000	4.0000	9.0000
1.1			4.9500	10.0000
1.2	6.0001	11.0002	6.0006	11.0000