SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 07

DUE ON: 04/02/'08

QUESTIONS:

→ Use modified Euler's method with the specified step size to determine the solution to the given IVP at the specified point.

(1) $y' = 4y - 1$; $y(0) = 1$; $y(0.5)$ with $h = 0.1$

(2) $y' = x - y^2$; $y(0) = 2$; $y(0.5)$ with $h = 0.1$

→ Apply RK4 with $h = 0.1$ to determine an approximation to the solution to the IVP

(3) $y' = y - x$; $y(0) = 0.5$; $y(0.5)$

(4) $y' = 2xy^2$; $y(0) = 0.5$; $y(0.5)$

→ (5) Approximate $y(0.2)$ where $y(t)$ is the solution of the IVP

$$y'' + 2y' + 4y = 0 \; ; \; y(0) = 2; \; \frac{dy(0)}{dt} = 0$$

using,

(a) Euler's method (b) RK4 method

with $h = 0.1$ for both methods. Find the exact solution of the problem and compare the actual value of $y(0.2)$ with y_2.

SOLUTIONS:

Modified Euler's Method:

(1) $y' = 4y - 1$; $y(0) = 1$; $y(0.5)$ with $h = 0.1$

Sol:- Given $y(0) = 1 \Rightarrow x_0 = 0$ & $y_0 = 1$

$$f(x_n, y_n) = 4y_n - 1 \quad ; h = 0.1$$

$$x_{n+1} = x_n + h$$

$n=0$: $x_1 = 0.1$; $y_1^* = y_0 + h f(x_0, y_0) = 1 + (0.1)(3) = 1.3$

$$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}\right] = 1 + (0.1)\left[\frac{3 + 4.2}{2}\right] = 1.36$$

$n=1$: $x_2 = 0.2$; $y_2^* = 1.36 + (0.1)(4.44) = 1.804$

$$y_2 = 1.36 + \left(\frac{0.1}{2}\right)\left[4.44 + 6.216\right] = 1.8928$$

$n=2$: $x_3 = 0.3$; $y_3^* = 1.8928 + (0.1)(6.5712) = 2.5499$

$$y_3 = 1.8928 + \left(\frac{0.1}{2}\right)\left[6.5712 + 9.1996\right] = 2.6813$$

$n=3$: $x_4 = 0.4$; $y_4^* = 2.6813 + (0.1)\left[9.7252\right] = 3.6538$

$$y_4 = 2.6813 + \left(\frac{0.1}{2}\right)\left[9.7252 + 13.6152\right] = 3.8483$$

$n=4$: $x_5 = 0.5$; $y_5^* = 3.8483 + (0.1)\left[14.3932\right] = 5.2876$

$$y_5 = 3.8483 + \left(\frac{0.1}{2}\right)\left[14.3932 + 20.1504\right] = 5.5755$$

$$\therefore \boxed{y(0.5) = 5.5755}$$

Modified Euler's Method:

(2) $y' = x - y^2$; $y(0) = 2$; $y(0.5)$ with $h = 0.1$

Sol:- Given $y(0) = 2 \Rightarrow x_0 = 0$; $y_0 = 2$; $h = 0.1$

$n=0$: $x_1 = 0.1$; $y_1^* = 2 + (0.1)[-4] = 1.6$

$$y_1 = 2 + \left(\frac{0.1}{2}\right)[-4 - 2.46] = 1.677$$

$n=1$: $x_2 = 0.2$; $y_2^* = 1.677 + (0.1)[-2.71237] = 1.4058$

$$y_2 = 1.677 + \left(\frac{0.1}{2}\right)[-2.7123 - 1.77763] = 1.4526$$

$n=2$: $x_3 = 0.3$; $y_3^* = 1.4526 + (0.1)[-1.9100] = 1.2616$

$$y_3 = 1.4526 + \left(\frac{0.1}{2}\right)[-1.91 - 1.2916] = 1.2925$$

$n=3$: $x_4 = 0.4$; $y_4^* = 1.2925 + (0.1)[-1.3706] = 1.1554$

$$y_4 = 1.2925 + \left(\frac{0.1}{2}\right)[-1.3706 - 0.9349] = 1.1772$$

$n=4$: $x_5 = 0.5$; $y_5^* = 1.1772 + (0.1)[-0.9858] = 1.0786$

$$y_5 = 1.1772 + \left(\frac{0.1}{2}\right)[-0.9858 - 0.6634] = 1.0947$$

$$\therefore \boxed{y(0.5) = 1.0947}$$

RK4 Method:

(3) $y' = y - x$; $y(0) = 0.5$; $y(0.5)$ with $h = 0.1$

Sol:- Given $y(0) = 0.5 \Rightarrow x_0 = 0$; $y_0 = 0.5$

$h = 0.1$; $f(x, y) = y - x$

For K_1, K_2, K_3, K_4:

$$K_1 = 0.1\, f(x_n, y_n) = 0.1\,(y_n - x_n)$$

$$K_2 = 0.1\, f(x_n + 0.05, y_n + 0.5K_1) = 0.1(y_n + 0.5K_1 - x_n - 0.05)$$

$$K_3 = 0.1\, f\left[x_n + 0.05,\ y_n + 0.5K_2\right] = 0.1\left[y_n + 0.5K_2 - x_n - 0.05\right]$$

$$K_4 = 0.1\, f\left[x_{n+1},\ y_n + K_3\right] = 0.1\left[y_n + K_3 - x_{n+1}\right]$$

$n=0$: $x_1 = 0.1$; $y_0 = 0.5$

$$K_1 = 0.1(0.5) = 0.05$$

$$K_2 = 0.1\left[0.5 + (0.5)(0.05) - 0.05\right] = 0.0475$$

$$K_3 = 0.1\left[0.5 + (0.5)(0.0475) - 0.05\right] = 0.0474$$

$$K_4 = 0.1\left[0.5 + 0.0474 - 0.1\right] = 0.0447$$

$$\therefore y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.5 + \frac{1}{6}(0.2845)$$

$$\Rightarrow y_1 = 0.5474.$$

$n=1$: $x_2 = 0.2$; $y_1 = 0.5474$

$$K_1 = 0.1\left[0.5474 - 0.1\right] = 0.0447$$

$$K_2 = 0.1\left[0.5474 + \tfrac{1}{2}(0.0447) - 0.1 - 0.05\right] = 0.0420$$

$$K_3 = 0.1\left[0.5474 + \tfrac{1}{2}(0.0420) - 0.1 - 0.05\right] = 0.0418$$

$$K_4 = 0.1\left[0.5474 + 0.0418 - 0.2\right] = 0.0389$$

$$y_2 = 0.5474 + \frac{1}{6}\left[0.0447 + 2(0.0420) + 2(0.0418) + 0.0389\right]$$

$$\Rightarrow y_2 = 0.5893$$

$n=2$: $x_3 = 0.3$; $y_2 = 0.5893$

$K_1 = 0.1(0.5893 - 0.2) = 0.0389$

$K_2 = 0.1\left[0.5893 + \frac{1}{2}(0.0389) - 0.2 - 0.05\right] = 0.0359$

$K_3 = 0.1\left[0.5893 + \frac{1}{2}(0.0359) - 0.2 - 0.05\right] = 0.0357$

$K_4 = 0.1\left[0.5893 + 0.0357 - 0.3\right] = 0.0325$

$\therefore y_3 = 0.6251$

$n=3$: $x_4 = 0.4$; $y_3 = 0.6251$

$K_1 = 0.1\left[0.6251 - 0.3\right] = 0.0325$

$K_2 = 0.1\left[0.6251 + \frac{1}{2}(0.0325) - 0.3 - 0.05\right] = 0.0291$

$K_3 = 0.1\left[0.6251 + \frac{(0.0291)}{2} - 0.3 - 0.05\right] = 0.0290$

$K_4 = 0.1\left[0.6251 + 0.0290 - 0.4\right] = 0.0254$

$\therefore y_4 = 0.6541$

$n=4$: $x_5 = 0.5$; $y_4 = 0.6541$

$K_1 = 0.1\left[0.6541 - 0.4\right] = 0.0254$

$K_2 = 0.1\left[0.6541 + \frac{1}{2}(0.0254) - 0.4 - 0.05\right] = 0.0217$

$K_3 = 0.1\left[0.6541 + \frac{1}{2}(0.0217) - 0.4 - 0.05\right] = 0.0215$

$K_4 = 0.1\left[0.6541 + 0.0215 - 0.5\right] = 0.0176$

$\therefore y_5 = 0.6756$

$\therefore \boxed{y(0.5) = 0.6756}$

RK4 Method:

(4) $y' = 2xy^2$; $y(0) = 0.5$; $y(0.5)$ with $h = 0.1$

Sol: Given $f(x,y) = 2xy^2$

$x_0 = 0$; $y_0 = 0.5$; $h = 0.1$

for K_1, K_2, K_3, K_4:

$$K_1 = 0.2\, x_n y_n^2$$

$$K_2 = 0.2\,(x_n + 0.05)\left(y_n + \frac{K_1}{2}\right)^2$$

$$K_3 = 0.2\,(x_n + 0.05)\left(y_n + \frac{K_2}{2}\right)^2$$

$$K_4 = 0.2\, x_{n+1}\,(y_n + K_3)^2$$

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$n=0$: $x_0 = 0$; $y_0 = 0.5$; ~~$x_1 = 0.1$~~ $x_1 = 0.1$

$$K_1 = 0.2(0)(0.5)^2 = 0$$

$$K_2 = 0.2(0 + 0.05)(0.5)^2 = 0.0025$$

$$K_3 = 0.2(0.05)\left(0.5 + \frac{0.0025}{2}\right)^2 = 0.0025$$

$$K_4 = 0.2(0.1)\left[0.5 + 0.0025\right]^2 = 0.0050$$

$$\therefore y_1 = 0.5025$$

$n=1$: $x_1 = 0.1$; $y_1 = 0.5025$; $x_2 = 0.2$

$$K_1 = (0.2)\left[(0.1)(0.5025)^2\right] = 0.005$$

$$K_2 = (0.2)\left[(0.1 + 0.05)\left(0.5025 + \frac{0.005}{2}\right)^2\right] = 0.0076$$

$$K_3 = (0.2)\left[(0.1 + 0.05)\left(0.5025 + \frac{0.0076}{2}\right)^2\right] = 0.0077$$

$$K_4 = (0.2)(0.2)(0.5025 + 0.0077)^2 = 0.0104$$

$$\therefore y_2 = 0.5102$$

$n=2$: $x_2 = 0.2$; $y_2 = 0.5102$; $x_3 = 0.3$

$$K_1 = (0.2)\left[(0.2)(0.5102)^2\right] = 0.0104$$

$$K_2 = (0.2)\left[(0.2+0.05)\left(0.5102 + \frac{0.0104}{2}\right)^2\right] = 0.0133$$

$$K_3 = (0.2)\left[(0.2+0.05)\left(0.5102 + \frac{0.0133}{2}\right)^2\right] = 0.0134$$

$$K_4 = (0.2)(0.3)\left(0.5102 + 0.0134\right)^2 = 0.0164$$

$$\therefore y_3 = 0.5236.$$

$n=3$: $x_3 = 0.3$; $y_3 = 0.5236$; $x_4 = 0.4$

$$K_1 = (0.2)\left[(0.3)(0.5236)^2\right] = 0.0164$$

$$K_2 = (0.2)\left[(0.3+0.05)\left(0.5236 + \frac{0.0164}{2}\right)^2\right] = 0.0198$$

$$K_3 = (0.2)\left[(0.3+0.05)\left(0.5236 + \frac{0.0198}{2}\right)^2\right] = 0.0199$$

$$K_4 = (0.2)(0.4)\left[0.5236 + 0.0199\right]^2 = 0.0236$$

$$\therefore y_4 = 0.5435.$$

$n=4$: $x_4 = 0.4$; $y_4 = 0.5435$; $x_5 = 0.5$

$$K_1 = (0.2)\left[(0.4)(0.5435)^2\right] = 0.0236$$

$$K_2 = (0.2)\left[(0.4+0.05)\left(0.5435 + \frac{0.0236}{2}\right)^2\right] = 0.0277$$

$$K_3 = (0.2)\left[(0.4+0.05)\left(0.5435 + \frac{0.0277}{2}\right)^2\right] = 0.0279$$

$$K_4 = (0.2)\left[(0.5)\left(0.5435 + 0.0279\right)^2\right] = 0.0326$$

$$\therefore y_5 = 0.5714$$

$$\therefore y(0.5) = 0.5714$$

(5) $y'' + 2y' + 4y = 0 \; ; \; y(0) = 2 \; ; \; \frac{dy(0)}{dt} = 0$

$y(0.2) = ? \qquad h = 0.1$

Sol:- (a) Euler's Method:

$$y'' + 2y' + 4y = 0$$

let $y' = v \Rightarrow y'' = v'$

$$\therefore v' + 2v + 4y = 0 \Rightarrow v' = -2v - 4y$$

$$y_{n+1} = y_n + h v_n$$

$$v_{n+1} = v_n + h\left[-2v_n - 4y_n\right]$$

$$x_0 = 0 \; ; \; y_0 = 2 \; ; \; v_0 = 0$$

$x = 0.1$:

$$y_1 = y_0 + h v_0 = 2 + 0 = 2$$

$$v_1 = v_0 + h\left[-2v_0 - 4y_0\right] = 0 + (0.1)(-8) = -0.8$$

$x = 0.2$:

$$y_2 = y_1 + h v_1 = 2 + (0.1)(-0.8) = 1.92$$

$$\therefore \boxed{y(0.2) = 1.92}$$

(b) RK4 Method:

$$y'' + 2y' + 4y = 0$$

let $y' = v \Rightarrow y'' = v'$

$$\therefore v' + 2v + 4y = 0 \Rightarrow v' = -2v - 4y$$

$$y_{n+1} = y_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$v_{n+1} = v_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

n=0:

$m_1 = v_0 = 0$

$m_2 = v_0 + \frac{1}{2}hK_1 = 0 + \frac{1}{2}(0.1)(-8) = -0.4$

$m_3 = v_0 + \frac{1}{2}hK_2 = 0 + \frac{1}{2}(0.1)(-7.2) = -0.36$

$m_4 = v_0 + hK_3 = 0 + (0.1)(-7.2) = -0.72$

$K_1 = -2(0) - 4(2) = -8$

$K_2 = -2\left[0 + \frac{1}{2}(0.1)(-8)\right] - 4\left[2 + \frac{1}{2}(0.1)(0)\right]$

$\Rightarrow K_2 = -7.2$

$K_3 = -2\left[0 + \frac{1}{2}(0.1)(-7.2)\right] - 4\left[2 + \frac{1}{2}(0.1)(-0.4)\right]$

$\Rightarrow K_3 = -7.2$

$K_4 = -2\left[0 + (0.1)(-7.2)\right] - 4\left[2 + (0.1)(-0.36)\right]$

$\Rightarrow K_4 = -6.416$

$\therefore y(0.1) = 1.9627$

$v(0.1) = -0.7203$

n=1:

$m_1 = -0.7203$

$m_2 = -0.7203 + \frac{1}{2}(0.1)(-6.4102)$

$\Rightarrow m_2 = -1.0408$

$m_3 = -0.7203 + \frac{1}{2}(0.1)(-5.6251)$

$\Rightarrow m_3 = -1.0015$

$m_4 = -0.7203 + (0.1)(-5.6396)$

$\Rightarrow m_4 = -1.2843$

$K_1 = -2(-0.7203) - 4(1.9627) = -6.4102$

$K_2 = -2(-1.0408) - 4\left[1.9627 + \frac{1}{2}(0.1)(-0.7203)\right]$

$\Rightarrow K_2 = -5.6251$

$K_3 = -2(-1.0015) - 4\left[1.9627 + \frac{1}{2}(0.1)(-1.0408)\right]$

$\Rightarrow K_3 = -5.6396$

$K_4 = -2(-1.2843) - 4\left[1.9627 + (0.1)(-1.0015)\right]$

$\Rightarrow K_4 = -4.8816$

$\therefore y(0.2) = 1.8612$

(c) Exact solution: $y'' + 2y' + 4y = 0 \; ; \; y(0) = 2 \; ; \; y'(0) = 0$

Let $y = e^{mt}$

$\therefore y' = me^{mt}$; $y'' = m^2 e^{mt}$

$\therefore e^{mt}\left[m^2 + 2m + 4\right] = 0$

$e^{mt} \neq 0 \Rightarrow m^2 + 2m + 4 = 0$

$\Rightarrow m = -1 \pm \sqrt{3}\, i$

$\therefore y(t) = e^{-t}\left[C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t\right]$

$y(0) = 2 \Rightarrow C_1 = 2$

$y'(0) = 0 \Rightarrow C_2 = \frac{2}{\sqrt{3}}$

$\therefore y(t) = e^{-t}\left[2\cos\sqrt{3}t + \frac{2}{\sqrt{3}}\sin\sqrt{3}t\right]$

$\therefore y(0.2) = 2e^{-0.2}\left[\cos\left(\sqrt{3} \times \frac{180}{\pi} \times 0.2\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3} \times \frac{180}{\pi} \times 0.2\right)\right]$

$\therefore \boxed{y(0.2) = 1.8612}$