SUB: MAE 3360

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SOLUTIONS TO ASSIGNMENT # 08

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QUESTIONS:

Find the inverse Laplace transform:

(1) $L^{-1}\left[\dfrac{3s+2}{(s-1)(s-2)}\right]$ (2) $L^{-1}\left[\dfrac{s-2}{(s+1)(s^2+4)}\right]$

(3) $L^{-1}\left[\dfrac{2s+3}{(s^2+4)(s^2+1)}\right]$ (4) $L^{-1}\left[\dfrac{s+4}{s^2+6s+13}\right]$

(5) $L^{-1}\left[\dfrac{s}{(s+1)^2+4}\right]$

Use Laplace transform to solve the given IVP.

(6) $y''+y'-2y=0$ ; $y(0)=1$ ; $y'(0)=4$

(7) $y''+5y'+4y=20\sin 2t$ ; $y(0)=1$ ; $y'(0)=-2$

(8) $y''-4y=2te^{t}$ ; $y(0)=0$ ; $y'(0)=0$

SOLUTIONS:

Find the inverse Laplace transform:

① $L^{-1}\left[\frac{3s+2}{(s-1)(s-2)}\right]$

Sol:- $\frac{3s+2}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$

$s=1$: $A=-5$

$s=2$: $B=8$

$\therefore \frac{3s+2}{(s-1)(s-2)} = \frac{-5}{s-1} + \frac{8}{s-2}$

$\therefore L^{-1}\left[\frac{3s+2}{(s-1)(s-2)}\right] = L^{-1}\left[\frac{-5}{s-1}\right] + L^{-1}\left[\frac{8}{s-2}\right] = 8e^{2t} - 5e^{t}.$

② $L^{-1}\left[\frac{s-2}{(s+1)(s^2+4)}\right]$

Sol:- $\frac{s-2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} \Rightarrow A(s^2+4)+(Bs+C)(s+1) = s-2$

$s=-1$: $A = \frac{-3}{5}$

$s=0$: $4A+C=-2 \Rightarrow C=\frac{2}{5}.$

Comparing the coefficients of $s^2$: $A+B=0$
$\Rightarrow B=\frac{3}{5}.$

$\therefore \frac{s-2}{(s+1)(s^2+4)} = \frac{-3}{5}\left[\frac{1}{s+1}\right] + \frac{3s+2}{5(s^2+4)}$

$$\therefore L^{-1}\left[\frac{s-2}{(s+1)(s^2+4)}\right] = \frac{-3}{5}L^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{5}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= -\frac{3}{5}e^{-t} + \frac{3}{5}\cos 2t + \frac{1}{5}\sin 2t.$$

(3) $L^{-1}\left[\dfrac{2s+3}{(s^2+4)(s^2+1)}\right]$

Sol:

$$\frac{2s+3}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow (As+B)(s^2+1) + (Cs+D)(s^2+4) = 2s+3$$

$$\Rightarrow (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D) = 2s+3$$

Comparing the coefficients and constants:

$s^3$: $A+C=0$ $\quad$ $s^2$: $B+D=0$

$s$: $A+4C=2$ $\quad$ Const: $B+4D=3$

On solving, $A = \frac{-2}{3}$; $B=-1$; $C=\frac{2}{3}$; $D=1$

$$\therefore \frac{2s+3}{(s^2+4)(s^2+1)} = \frac{-\frac{2}{3}s-1}{s^2+4} + \frac{\frac{2}{3}s+1}{s^2+1}$$

$$= \frac{-1}{3}\left[\frac{2s+3}{s^2+4}\right] + \frac{1}{3}\left[\frac{2s+3}{s^2+1}\right]$$

$$L^{-1}\left[\frac{2s+3}{(s^2+4)(s^2+1)}\right] = -\frac{2}{3}L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{3}{6}L^{-1}\left[\frac{2}{s^2+4}\right] + \frac{2}{3}L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{2}{3}\cos 2t - \frac{1}{2}\sin 2t + \frac{2}{3}\cos t + \sin t$$

④ $L^{-1}\left[\frac{s+4}{s^2+6s+13}\right]$

$$= L^{-1}\left[\frac{(s+3)+1}{(s+3)^2+4}\right] = L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right] + L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$$\therefore L^{-1}\left[\frac{s+4}{s^2+6s+13}\right] = e^{-3t}\cos 2t + \frac{1}{2}e^{-3t}\sin 2t.$$

$$\left\{\begin{aligned} L\left[e^{at}\cos bt\right] &= \frac{s-a}{(s-a)^2+b^2} \\ L\left\{e^{at}\sin bt\right] &= \frac{b}{(s-a)^2+b^2} \end{aligned}\right\}$$

⑤ $L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$

Sol:- $$\frac{s}{(s+1)^2+4} = \frac{(s+1)-1}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}$$

$$\therefore L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{e^{-t}}{2}\left[2\cos 2t - \sin 2t\right]$$

→ Use Laplace transform to solve the given IVP:

⑥ $y'' + y' - 2y = 0$ ; $y(0) = 1$ ; $y'(0) = 4$

Sol:- $y'' + y' - 2y = 0$

applying Laplace transform on both sides,

$$L[y''] + L[y'] - 2L[y] = L[0]$$

$$\Rightarrow \left[s^2 L[Y] - s\,y(0) - y'(0)\right] + \left[s L[Y] - y(0)\right] - 2L[Y] = L[0]$$

$$y(0) = 1 \ ; \ y'(0) = 4.$$

$$\therefore \left[s^2 L[Y] - s - 4\right] + s L[Y] - 1 - 2L[Y] = 0$$

$$\Rightarrow (s^2+s-2)\,L[Y] - s - 5 = 0$$

$$\Rightarrow Y(s) = \frac{s+5}{s^2+s-2}$$

$$\frac{s+5}{s^2+s-2} = \frac{s+5}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$s = 1$ : $A = 2$

$s = -2$ : $B = -1$

$$\therefore \frac{s+5}{(s-1)(s+2)} = \frac{2}{s-1} - \frac{1}{s+2}$$

$$\therefore Y(s) = \frac{2}{s-1} - \frac{1}{s+2}$$

applying inverse laplace transforms on both sides

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$\Rightarrow y(t) = 2e^{t} - e^{-2t}$$

(7) $y'' + 5y' + 4y = 20\sin 2t \ ; \ y(0) = 1 \ ; \ y'(0) = -2$

Sol:- Given $y'' + 5y' + 4y = 20\sin 2t$.

applying laplace transform on both sides,

$$L[y''] + 5L[y'] + 4L[y] = 20\,L[\sin 2t]$$

$$\Rightarrow \left[s^2 L[Y] - s y(0) - y'(0)\right] + 5\left[s L[Y] - y(0)\right] + 4 L(Y) = 20 L[\sin 2t]$$

$$\Rightarrow \left[s^2 L[Y] - s + 2\right] + 5\left[s L[Y] - 1\right] + 4 L[Y] = \frac{40}{s^2+4}$$

$$\Rightarrow (s^2+5s+4) L[Y] - s + 2 - 5 = \frac{40}{s^2+4}$$

$$\Rightarrow (s^2+5s+4) L[Y] = \frac{40}{s^2+4} + s + 3$$

$$\Rightarrow Y(s) = \frac{40}{(s^2+4)(s+1)(s+4)} + \frac{s+3}{(s+1)(s+4)}$$

$$\Rightarrow Y(s) = \frac{40 + (s+3)(s^2+4)}{(s^2+4)(s+1)(s+4)}$$

$$\frac{40 + (s+3)(s^2+4)}{(s^2+4)(s+1)(s+2)} = \frac{As+B}{s^2+4} + \frac{C}{s+1} + \frac{D}{s+2}$$

$s=-1$: $C = 10/3$

$s=-4$: $D = -1/3$

$s=0$: $B = 0$

Comparing the Coefficients of $s^3$:

$$A + C + D = 1$$
$$\Rightarrow A = -2$$

$$\therefore \frac{40 + (s+3)(s^2+4)}{(s^2+4)(s+1)(s+2)} = \frac{-2s}{s^2+4} + \frac{(10/3)}{s+1} - \frac{(1/3)}{s+2}$$

$$\therefore Y(s) = \frac{-2s}{s^2+4} + \frac{10/3}{s+1} - \frac{(1/3)}{s+2}$$

— applying inverse Laplace transform on both sides,

$$L^{-1}[Y(s)] = -2L^{-1}\left(\frac{s}{s^2+4}\right] + \frac{10}{3}L^{-1}\left(\frac{1}{s+1}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s+4}\right]$$

$$\therefore y(t) = -2\cos 2t + \frac{10}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

(8) $y'' - 4y = 2te^{t}$ ; $y(0) = 0$ ; $y'(0) = 0$

Sol:- Given $y'' - 4y = 2te^{t}$

applying Laplace transforms on both sides,

$$L[y''] - 4L[y] = 2L[te^{t}]$$

$$\Rightarrow \left[s^2 L[y] - s y(0) - y'(0)\right] - 4L[y] = 2L[te^{t}]$$

$\Rightarrow$ $y(0) = 0$ ; $y'(0) = 0$

$$\left\{\because L[te^{t}] = \frac{1}{(s-1)^2}\right\}$$

$$\therefore (s^2 - 4)\, L[y] = \frac{2}{(s-1)^2}$$

$$\therefore L[y] = \frac{2}{(s-1)^2(s+2)(s-2)}$$

$$\frac{2}{(s-1)^2(s+2)(s-2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} + \frac{D}{s-2}$$

$s = 1$: $B = -2/3$

$s = -2$: $C = -1/18$

$s = 2$: $D = 1/2$

Coefficients of $s^3$: $A + C + D = 0 \Rightarrow A = -\frac{4}{9}$

$$\therefore L[y] = Y(s) = \frac{-4/9}{s-1} + \frac{(-2/3)}{(s-1)^2} + \frac{(-1/18)}{s+2} + \frac{1/2}{s-2}$$

applying inverse Laplace transform on both sides,

$$L^{-1}\{Y(s)\} = -\frac{4}{9}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{18}L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s-2}\right]$$

$$\Rightarrow y(t) = -\frac{4}{9}e^{t} - \frac{2}{3}te^{t} - \frac{1}{18}e^{-2t} + \frac{1}{2}e^{2t}.$$

(or) $$y(t) = -\frac{2}{9}e^{t}\left[2+3t\right] - \frac{1}{18}e^{-2t} + \frac{1}{2}e^{2t}$$

(or) $$y(t) = -\frac{4}{9}e^{t} - \frac{2}{3}te^{t} + \frac{4}{9}\cosh 2t + \frac{5}{9}\sinh 2t$$