MAE 3360

HOMEWORK # 10

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Exercises 9.9 - Independence of Path

2. a) $\int_{(1,1)}^{(2,4)} 2xy\,dx + x^2 dy$

$P_y = \cancel{2y}\ 2x$

$Q_x = 2x$

and the integral is independent of path

$\phi_x = 2xy$

$\phi = x^2 y + g(y)$

$\phi_y = x^2 + g'(y) = x^2$

$\therefore g(y) = 0$

$\therefore \phi = x^2 y$

$$\int_{(1,1)}^{(2,4)} 2xy\,dx + x^2 dy = \left[x^2 y\right]_{(1,1)}^{(2,4)} = 16 - 1 = 15$$

b.) $y = 3x - 2$ for $1 \leq x \leq 2$

$$\int_{(1,1)}^{(2,4)} 2xy\,dx + x^2 dy = \int_1^2 [2x(3x-2) + x^2(3)]\,dx$$

$$= \int_1^2 (9x^2 - 4x)\,dx = \left[3x^3 - 2x^2\right]_1^2 = 15$$

4.a. $\displaystyle\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y \, dx + (1-\sin x \sin y)\, dy$

$P_y = -\cos x \sin y$
$Q_x = -\cos x \sin y$
$\therefore$ The integral is independent of path.

$\phi_x = \cos x \cos y$
$\phi = \sin x \cos y + g(y)$
$\phi_y = -\sin x \sin y + g'(y) = 1 - \sin x \sin y$

$\therefore g(y) = y$
$\therefore \phi = \sin x \cos y + y$

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y \, dx + (1-\sin x \sin y)\, dy$$
$$= \Big[(\sin x \cos y + y)\Big]_{(0,0)}^{(\pi/2,0)} = 1$$

b) Use $y=0$ for $0 \le x \le \pi/2$

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y \, dx + (1-\sin x \sin y)\, dy = \int_0^{\pi/2} \cos x \, dx$$
$$= \Big[\sin x\Big]_0^{\pi/2} = 1$$

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12. $F(x,y) = 2xy^3 i + 3y^2(x^2+1) j$

$P_y = 6xy^2 = Q_x$

$\therefore$ The vector field is a gradient field.

$\phi_x = 2xy^3$

$\phi = x^2y^3 + g(y)$

$\phi_y = 3x^2y^2 + g'(y) = 3x^2y^2 + 3y^2$

$\therefore g(y) = y^3$

$\therefore \phi = x^2y^3 + y^3$

16. $F(x,y) = 2e^{2y} i + xe^{2y} j$

$P_y = 4e^{2y}$

$Q_x = e^{2y}$

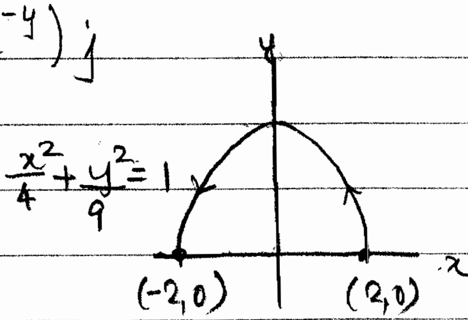
$\therefore$ The vector field is not a gradient field.

18. $F(x,y) = (2x + e^{-y}) i + (4y - xe^{-y}) j$

$P_y = -e^{-y} = Q_x$

$\therefore F$ is conservative

and $\int_C F \cdot dr$ is independent of path



Hence, the work done can be evaluated along the curve: $y=0$, $-2 \leq x \leq 2$

Then $dy=0$

and $W = \int (2x+e^{-y})dx + (4y - xe^{-y})dy$

$$= \int_{2}^{-2} (2x+1)\,dx = \left[x^2+x\right]_{2}^{-2}$$

$$= -4$$

20. $\displaystyle\int_{(0,0,0)}^{(1,1,1)} 2x\,dx + 3y^2 dy + 4z^3 dz$

$P_y = 0 = Q_x$

$Q_z = 0 = R_y$

$R_x = 0 = P_z$

$\therefore$ The integral is independent of path.

Parameterize the line segment between the points by $x=t$, $y=t$, $z=t$ for $0 \leq t \leq 1$

$\therefore dx = dy = dz = dt$

$$\int_{(0,0,0)}^{(1,1,1)} 2x\,dx + 3y^2 dy + 4z^3 dz = \int_0^1 (2t + 3t^2 + 4t^3)\,dt$$

$$= (t^2 + t^3 + t^4)\Big|_0^1 = 3$$

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21. $\int_{(1,0,0)}^{(2,\pi/2,1)} (2x\sin y + e^{3z})dx + x^2\cos y\,dy + (3xe^{3z}+5)dz$

$P_y = 2x\cos y = Q_x$

$Q_z = 0 = R_y$

$R_x = 3e^{3z} = P_z$

$\therefore$ The integral is independent of path

$\therefore \phi_x = 2x\sin y + e^{3z}$

$\phi = x^2\sin y + xe^{3z} + g(y,z)$

$\phi_y = x^2\cos y + g(y) = Q = x^2\cos y$

$\therefore g_y = 0$

$g(y,z) = h(z) \quad \therefore$

$\therefore \phi = x^2\sin y + xe^{3z} + h(z)$

$\phi_z = 3xe^{3z} + h'(z) = R = 3xe^{3z} + 5$

$\therefore h'(z) = 5$

$\therefore h(z) = 5z$

$\therefore \phi = x^2\sin y + xe^{3z} + 5z$

$\therefore \int_{(1,0,0)}^{(2,\pi/2,1)} (2x\sin y + e^{3z})dx + x^2\cos y\,dy + (3xe^{3z}+5)dz$

$= \left[x^2\sin y + xe^{3z} + 5z\right]_{(1,0,0)}^{(2,\pi/2,1)} = 8 + 2e^3$

28. $F(x,y,z) = 8xy^3z\,i + 12x^2y^2z\,j + 4x^2y^3k$

$r(t) = 2\cos t\,i + 2\sin t\,j + tk$

$P_y = 24xy^2z = Q_x$

$Q_z = 12x^2y^2 = R_y$

$R_x = 8xy^3 = P_z$

$\therefore$ F is conservative

Thus, the work done between two points is independent of path.

$\phi_x = 8xy^3z$

$\phi = 4x^2y^3z$

which is a potential function of F

$$\therefore W = \int_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} F\cdot dr = \left[4x^2y^3z\right]_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} = 4\sqrt{3}\pi$$

$$\& \; W = \int_{(2,0,0)}^{(0,2,\pi/2)} F\cdot dr = 0$$