Worked Examples

Part IV Fourier Series and Partial Differential Equations

12 Orthogonal Functions and Fourier Series

EXERCISES 12.1

Orthogonal Functions

1. $\int_{-2}^{2} x x^2 dx = \frac{1}{4}x^4 \Big|_{-2}^{2} = 0$

2. $\int_{-1}^{1} x^3(x^2+1)dx = \frac{1}{6}x^6 \Big|_{-1}^{1} + \frac{1}{4}x^4 \Big|_{-1}^{1} = 0$

3. $\int_{0}^{2} e^x(xe^{-x} - e^{-x})dx = \int_0^2 (x-1)dx = \left(\frac{1}{2}x^2 - x\right)\Big|_0^2 = 0$

4. $\int_0^{\pi} \cos x \sin^2 x\, dx = \frac{1}{3}\sin^3 x \Big|_0^{\pi} = 0$

5. $\int_{-\pi/2}^{\pi/2} x\cos 2x\, dx = \frac{1}{2}\left(\frac{1}{2}\cos 2x + x\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} = 0$

6. $\int_{\pi/4}^{5\pi/4} e^x \sin x\, dx = \left(\frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x\right)\Big|_{\pi/4}^{5\pi/4} = 0$

7. For $m \neq n$

$$\int_0^{\pi/2} \sin(2n+1)x\, \sin(2m+1)x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \Big(\cos 2(n-m)x - \cos 2(n+m+1)x\Big)\, dx$$

$$= \frac{1}{4(n-m)}\sin 2(n-m)x \Big|_0^{\pi/2} - \frac{1}{4(n+m+1)}\sin 2(n+m+1)x \Big|_0^{\pi/2} = 0.$$

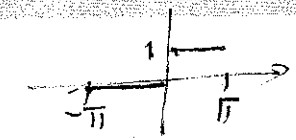
25. In R^3 the set $\{\mathbf{i}, \mathbf{j}\}$ is not complete since $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete. To see this suppose that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then

$$0 = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \mathbf{i}) = a(\mathbf{i}, \mathbf{i}) + b(\mathbf{j}, \mathbf{i}) + c(\mathbf{k}, \mathbf{i}) = a(1) + b(0) + c(0) = a.$$

Similarly, $b = 0$ and $c = 0$. Thus, the only vector in R^3 orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is $\mathbf{0}$, so $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete.

EXERCISES 12.2

Fourier Series

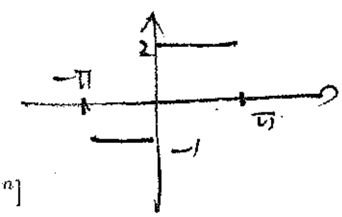


1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}[1 - (-1)^n]$

$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

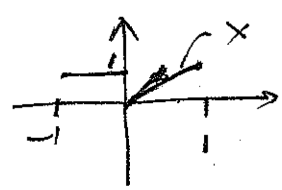


2. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_0^{\pi} 2\,dx = 1$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\cos nx\,dx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\sin nx\,dx = \frac{3}{n\pi}[1 - (-1)^n]$

$f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

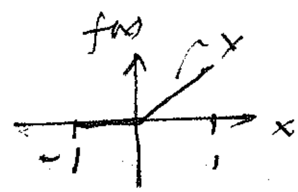


3. $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^1 x\,dx = \frac{3}{2}$

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_0^1 x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\sin n\pi x\,dx + \int_0^1 x\sin n\pi x\,dx = -\frac{1}{n\pi}$

$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$



4. $a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^1 x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^1 x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$