MAE 3360

EXAM - II - Review (I)

Dt: 04/14/2008

① **Variation of Parameter method:**

For a given ordinary differential equation, we can find the solution of ODE using variation of Parameter method as follows.

(i) Homogeneous solution: (y_h):

It can be find for given ODE

~~$L_n(y)=r$~~ $L_n(y) = r(x)$

take $L_n(y)=0$ and then we can solve it using Characteristic equation.

$$\boxed{y_h = C_1 y_1(x) + C_2 y_2(x)}$$

(ii) Particular Solution: (y_p)

Let, $y_p = u(x)\, y_1 + v(x)\, y_2$.

Now,

$$\left.\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1' + v' y_2' &= r(x) \end{aligned}\right\}$$

We can solve these simultaneous equations by using Cramer's rule.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Solving for u' & v'

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix}}{W} \quad \& \quad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix}}{W}$$

Now we can find $u = \int u' \, dx$ & $v = \int v' \, dx$

then substitute these values to the original y_p

(iii) General solution:-

Sum the y_p & y_h.

$$\therefore \boxed{y = y_h + y_p}$$

Ex:- Solve $y'' + 4y = \sin^2 2t$.

Solution: Homogeneous solution:-

char. eq^n: $m^2 + 4 = 0$.

$\Rightarrow m = \pm 2i$

$$\Rightarrow y_h(x) = c_1 \underset{\uparrow \, y_1}{\cos 2t} + c_2 \underset{\uparrow \, y_2}{\sin 2t}$$

Particular Solution:-

$R(x) = \sin^2 2t$

Let $y_p = u\, y_1 + v\, y_2$

$y_p = u \cos 2t + v \sin 2t$

Now, $u' y_1 + v' y_2 = 0$

$\therefore u' \cos 2t + v' \sin 2t = 0$ — (i)

$u' y_1' + v' y_2' = 0$

$\therefore -2u' \sin 2t + 2v' \cos 2t = 0$ — (ii) solving (i) & (ii)

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t$$

$$= 2\,(1)$$

$$\boxed{W = 2}$$

using cramer's rule,

$$u' = \frac{\begin{vmatrix} 0 & \sin 2t \\ \sin^2 2t & 2\cos 2t \end{vmatrix}}{2} = -\frac{1}{2}\sin^3 2t$$

$$v' = \frac{\begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \sin^2 2t \end{vmatrix}}{2} = \frac{1}{2}\left(\sin^2 2t \cos 2t\right)$$

Now, $u = \int u'\,dt = \int -\frac{1}{2}\sin^3 2t \, dt$

$$u = \frac{1}{4}\cos 2t - \frac{1}{12}\cos^3 2t$$

$\therefore v = \int v'\,dt = \int \frac{1}{2}\sin^2 2t \cdot \cos 2t \, dt$

$$v = \frac{1}{12}\sin^3 2t$$

Now, $y_p = u \cdot y_1 + v\, y_2$

$$= \left(\frac{1}{4}\cos 2t - \frac{1}{12}\cos^3 2t\right)\cos 2t + \left(\frac{1}{12}\sin^3 2t\right)\sin 2t$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\left(\cos^4 2t - \sin^4 2t\right)$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\left(\cos^2 2t - \sin^2 2t\right)\underbrace{\left(\cos^2 2t + \sin^2 2t\right)}_{1}$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\cos^2 2t + \frac{1}{12}\sin^2 2t$$

$$y_p = \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t$$

Now, $y = y_h + y_p$

$$\boxed{y = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t}$$

solution.

Note: If you have given the Initial conditions then you can find C_1 & C_2 from it and substitute to the final answer.

(4) Spring / mass System:

(1) Free Undamped motion:

For free undamped motion the equation of system is given, as

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (\because \omega^2 = k/m)$$

The solution of this equation,

$$\boxed{x(t) = C_1 \cos\omega t + C_2 \sin\omega t}$$ - Equation of motion.

We can write this equation in the form

$$X(t) = A\sin(\omega t + \phi) \qquad \text{where, } A = \sqrt{C_1^2 + C_2^2}, \quad \phi = \tan^{-1}(C_1/C_2)$$

(2) Free damped motion:

For free damped motion the equation of system is defined as...

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0 \qquad \beta = \text{damping factor}$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \qquad \text{where, } 2\lambda = \beta/m, \quad \omega^2 = k/m$$

Now the solution of the above equation is known as equation of motion for such system. There are 3 cases for this.

(i) $\lambda^2 - \omega^2 > 0$.

$$\Rightarrow x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

Such a system called Overdamped system.

(ii) $\lambda^2 - \omega^2 = 0$.

$$\Rightarrow x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

Such a system called critically damped system.

(iii) $\lambda^2 - \omega^2 < 0$.

$$\Rightarrow x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t\right)$$

Such a system known as Underdamped system.

Ex: $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$; $y(0) = -1$ & $y'(0) = 5$.

Find the equation of motion for given system and find whether its overdamped, underdamped or critically damped.

Solution: The system of equation is

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$$

Comparing with $\frac{d^2y}{dx^2} + 2\lambda\frac{dy}{dx} + \omega^2 y = 0.$

$\Rightarrow 2\lambda = 7.$ & $\omega^2 = 12.$

Now, $\lambda^2 = \frac{49}{4}.$

$$\lambda^2 - \omega^2 = \frac{49}{4} - 12 = \frac{1}{4} > 0$$

So, its $\boxed{\text{Overdamped}}$ system.

equation of motion is

$$y(x) = e^{-\lambda x}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,x} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,x}\right)$$

$$= e^{-\frac{7}{2}x}\left(c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}\right)$$

$$= e^{-\frac{7}{2}x}\left(c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}\right)$$

$\therefore y(x) = c_1 e^{-3x} + c_2 e^{-4x}$

$\Rightarrow y'(x) = -3c_1 e^{-3x} - 4c_2 e^{-4x}$

$y'(0) = 5 = -3c_1 - 4c_2$

$y(0) = -1 = c_1 + c_2$

solving eq^ns we have $c_2 = -2$ & $c_1 = 1$

$$\therefore \boxed{y(x) = e^{-3x} - 2e^{-4x}}$$ ← equation of motion.

(3) Forced driven system:

If there is some external force $f(t)$ required for the motion of spring-mass system, then the equation of system is described as..

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)} ; \quad 2\lambda = \beta/m, \quad \omega^2 = k/m, \quad F(t) = f(t)/m$$

We can solve this equation by using the method of undetermined co-efficient or variation of parameter method.

(4) Undamped forced motion: If there will be no damping force in the forced driven system, then the equation of system would be,

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = F(t)}$$

We can solve is as same as for forced driven system.

Note: See SI notes from March 5 to March 14 for the reference examples.