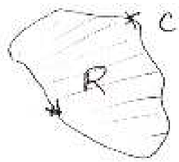
SI Session.

MAE 3360

Dt: 04/28/2008

SI leader: Monalkumar Patel.

⊛ Green's theorem:



If any region R is closed by contour 'c' then green's theorem defined as relation between the area integral and Contour (cyclic) Integral.

$$\boxed{\oint_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}$$

→ P, Q, $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are continuous inside the region and also upon the boundary 'c'.

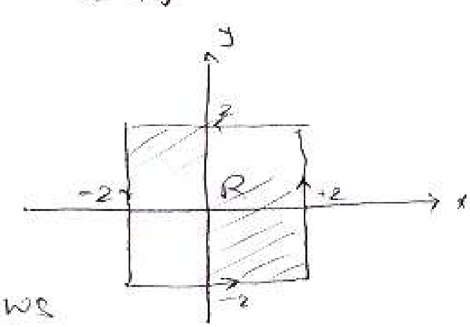
→ In general the green's theorem is defined as

$$\oint \vec{v} \cdot d\vec{r} = \iint (\nabla \times \vec{v}) \cdot \vec{n}\, d\vec{r}$$

→ If P, Q, $\frac{\partial Q}{\partial x}$ or $\frac{\partial P}{\partial y}$ is not continuous within the region R then Green's Theorem is NOT applicable.

e.g. $\oint_c \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

'c' is defined as...

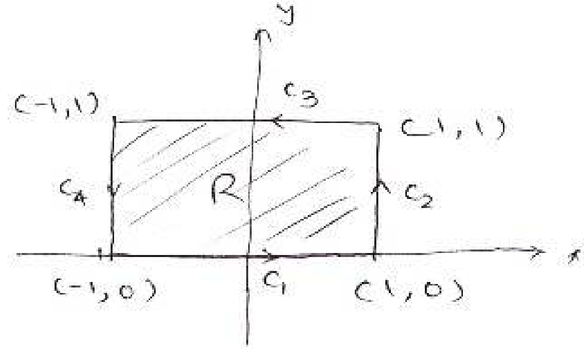


Here, Integral shows singularity about 'o' (origin) so, Green's theorem is not applicable.

Ex: (2) Verify the green's theorem:

$$\oint_C 3x^2y\,dx + (x^2-5y)\,dy = \iint_R (2x-3x^2)\,dA$$ where 'c' is the rectangle with vertices $(-1,0)$, $(1,0)$, $(1,1)$ and $(-1,1)$.

Solution:. The Integral path 'c' is graphically defined as follows...



→ Now we devided Integral path 'c' in c_1, c_2, c_3 & c_4.

Now, $\oint_C 3x^2y\,dx + (x^2-5y)\,dy$

$$\oint = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4}$$

Along c_1: $(-1,0) \to (1,0)$

$\Rightarrow y=0 \Rightarrow dy = 0$

So, $\oint_{c_1} 3x^2y\,dx + (x^2-5y)\,dy$

$$= \int_{-1}^{1} 0\,dx = [c]_{-1}^{1} = 0$$

Along c_2: $(1,0) \to (1,1)$ $\quad x=1 \Rightarrow dx=0 \quad y: 0 \to 1$

So, $\int_{c_2} 3x^2y\,dx + (x^2-5y)\,dy = \int_0^1 0 + (1-5y)\,dy$

$$= \left[y - \frac{5}{2}y^2\right]_0^1$$

$$= -3/2$$

Along C_3: $(1,1) \to (-1,1)$

$$y = 1 \Rightarrow dy = 0 \qquad x : 1 \to -1$$

$$\therefore \int_{C_3} 3x^2y\,dx + (x^2-5y)\,dy$$

$$= \int_1^{-1} 3x^2\,dx$$

$$= \left[x^3\right]_1^{-1} = -2$$

Along C_4: $(-1,1) \to (-1,0)$

$$\Rightarrow x = -1 \Rightarrow dx = 0 \qquad y : 1 \to 0$$

$$\therefore \int_{C_4} 3x^2y\,dx + (x^2-5y)\,dy = \int_1^0 [1-5y]\,dy$$

$$= \left[y - \frac{5}{2}y^2\right]_1^0$$

$$= -1 + 5/2$$

$$= 3/2$$

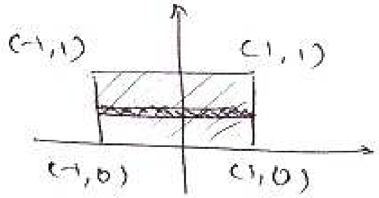
Now, $$\oint_C 3x^2y\,dx + (x^2-5y)\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= 0 - \frac{3}{2} - 2 + 3/2$$

$$\boxed{\therefore \oint_C 3x^2y\,dx + (x^2-5y)\,dy = -2} \quad \text{—— (II)}$$

Now, $$\iint_R (2x - 3x^2)\,dA$$

$$dA = dx\,dy$$



$$\Rightarrow \iint_R (2x - 3x^2)\,dA$$

$$= \int_0^1 \int_{-1}^1 (2x - 3x^2)\,dx\,dy$$

$$= \int_0^1 [x^2 - x^3]_{-1}^{1} \, dy$$

$$= \int_0^1 -2 \, dy$$

$$= [-2y]_0^1 = -2$$

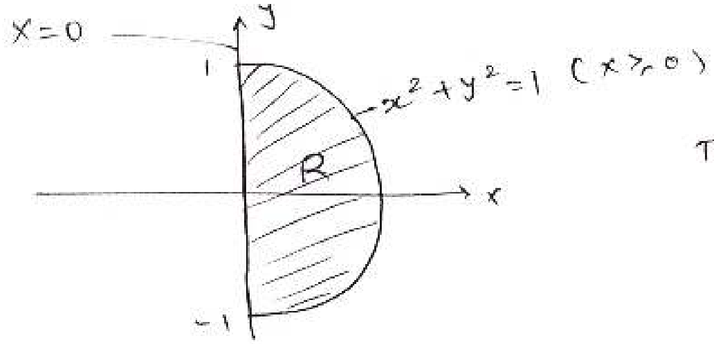
$$\therefore \boxed{\iint_R (2x - 3x^2) \, dA = -2} \quad \text{--- (2)}$$

from (1) & (2) we can verify the green's theorem

$$\boxed{\oint_C (3x^2y \, dx + (x^2 - 5y) \, dy = \iint_R (2x - 3x^2) \, dA}$$

Ex:- (11) calculate $\oint_C xy \, dx + x^2 \, dy$ where 'c' is the boundary of the region determined by the graphs of $x = 0$, $x^2 + y^2 = 1$, $x \geq 0$.

Solution:- For the given conditions the 'c' is defined as follows...



The region 'R' is as shown in figure

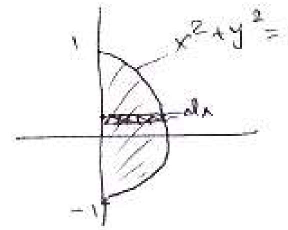
So, using Green's theorem,

$$\oint_C xy \, dx + x^2 \, dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right] dA$$

$$= \iint_R (2x - x)\, dA$$

$$= \iint_R x\, dA$$

Now from figure the limits are defined as



$$= \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} x \, dx \, dy$$

$$= \int_{-1}^{1} \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - y^2)\, dy$$

$$= \frac{1}{2} \left[y - \frac{y^3}{3} \right]_{-1}^{1} = \frac{1}{2} \left[1 - \frac{1}{3} + 1 - 1/3 \right] = 1 - 1/3$$

$$= 2/3$$

$$\boxed{\therefore \oint_c xy\, dx + x^2\, dy = 2/3}$$

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Dt: 04/30/2008

(*) Orthogonal functions:-

The two functions $f_1(x)$ and $f_2(x)$ can be said orthogonal to each other over an interval $[a,b]$ if...

$$\boxed{\int_a^b f_1(x)\, f_2(x)\, dx = 0}$$

e.g.

$\{1, \cos x, \cos 2x, \ldots, \cos mx, \ldots\}$ is called a set of orthogonal functions over an interval $[-\pi, \pi]$

Similarly,

$\{\sin x, \sin 2x, \sin 3x, \ldots \sin mx, \ldots\}$ is also orthogonal set over $[-\pi, \pi]$.

Ex: (5) $f_1(x) = x$ & $f_2(x) = \cos 2x$ check that they are orthogonal over ~~set~~ $[-\pi/2, \pi/2]$ or not.

Solution:- $f_1(x) = x$ & $f_2(x) = \cos 2x$.

So, from definition,

$$\int_{-\pi/2}^{\pi/2} f_1(x)\, f_2(x)\, dx$$

$$= \int_{-\pi/2}^{\pi/2} x \cos 2x \, dx$$

using Integration by Parts...

$$= x \cdot \frac{\sin 2x}{2} \Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{2} \, dx$$

$$= 0 - \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin 2x \, dx$$

$$= +\frac{1}{2} \left[\frac{\cos 2x}{2} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\cos(\pi) - \cos(-\pi)}{4}.$$

$$= 0$$

$$\therefore \boxed{\int_{-\pi/2}^{\pi/2} f_1(x) \, f_2(x) = 0}$$

So, $f_1(x) = x$ & $f_2(x) = \cos 2x$ are orthogonal over $[-\pi/2, \pi/2]$

Note: $\{1, \cos \frac{\pi x}{p}, \cos \frac{2\pi x}{p}, \cos \frac{3\pi x}{p}, \ldots \cos \frac{m\pi x}{p}, \ldots$

$\sin \frac{\pi x}{p}, \sin \frac{2\pi x}{p}, \ldots \sin \frac{m\pi x}{p}, \ldots \}$ is orthogonal

set over $[-p, p]$. (You can verify using similar way)

Fourier Series:-

In general the fourier series of function f(x) over given interval is defined as... [-P, P] is interval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{P}\right) + b_n \sin\left(\frac{n\pi x}{P}\right) \right]$$

where,

$$a_0 = \frac{1}{P} \int_{-P}^{P} f(x)\, dx$$

$$a_n = \frac{1}{P} \int_{-P}^{P} f(x) \cos\left(\frac{n\pi x}{P}\right) dx$$

$$b_n = \frac{1}{P} \int_{-P}^{P} f(x) \sin\left(\frac{n\pi x}{P}\right) dx.$$

Note: If the given interval is $[-\pi, \pi]$ for f(x) then...

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

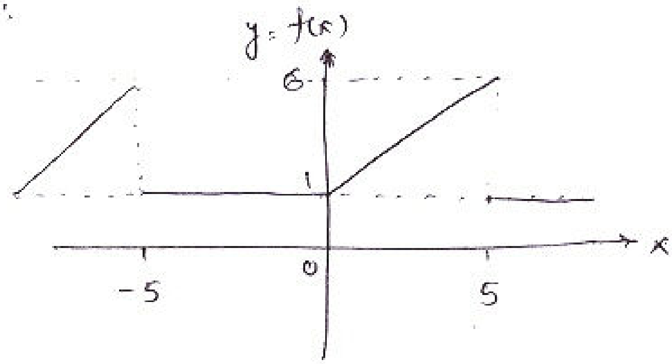
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Ex:- (13) Find the fourier series for

$$f(x) = \begin{cases} 1 & ; -5 < x < 0 \\ 1+x & ; 0 \leq x < 5 \end{cases}$$

Solution:



Now, the Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right] \quad - (*)$$

Here $[-p, p] = [-5, 5]$

So, $a_0 = \frac{1}{5} \int_{-5}^{5} f(x)\, dx$

$$= \frac{1}{5} \left[\int_{-5}^{0} 1\, dx + \int_{0}^{5} (1+x)\, dx \right]$$

$$= \frac{1}{5} \left[x \Big|_{-5}^{0} + \left(x + \frac{x^2}{2}\right) \Big|_{0}^{5} \right]$$

$$= \frac{1}{5} \left[5 + 5 + \frac{25}{2} \right]$$

$$\boxed{a_0 = 9/2}$$

Now, $$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 1\cdot\cos\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{5}(1+x)\cos\left(\frac{n\pi x}{5}\right)dx\right]$$

$$= \frac{1}{5}\left[\int_{-5}^{5}\cos\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{5} x\cdot\cos\left(\frac{n\pi x}{5}\right)dx\right]$$

$$= \frac{1}{5}\left[\frac{\sin\left(\frac{n\pi x}{5}\right)}{(n\pi/5)}\Bigg|_{-5}^{5} + x\cdot\frac{\sin\left(\frac{n\pi x}{5}\right)}{\left(\frac{n\pi}{5}\right)}\Bigg|_{0}^{5} - \int_{0}^{5}\frac{\sin\left(\frac{n\pi x}{5}\right)}{(n\pi/5)}dx\right]$$

$$= \frac{1}{5}\left[0 + 0 + \frac{\cos\left(\frac{n\pi x}{5}\right)}{(n\pi/5)^2}\Bigg|_{0}^{5}\right]$$

$$= \frac{5}{(n\pi)^2}\left[\cos(n\pi) - \cos(0)\right]$$

$$\boxed{a_n = \frac{5}{n^2\pi^2}\left[(-1)^n - 1\right]}$$

Now, $$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 1\cdot\sin\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{5}(1+x)\sin\left(\frac{n\pi x}{5}\right)dx\right]$$

$$= \frac{1}{5}\left[\int_{-5}^{5}\sin\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{5} x\cdot\sin\left(\frac{n\pi x}{5}\right)dx\right]$$

$$= \frac{1}{5}\left[\left.\frac{\cos\left(\frac{n\pi x}{5}\right)}{(n\pi/5)}\right|_{-5}^{5} - x \cdot \left.\frac{\cos\left(\frac{n\pi x}{5}\right)}{(n\pi/5)}\right|_{0}^{5} - \int_{0}^{5} \frac{\cos\left(\frac{n\pi x}{5}\right)}{(n\pi/5)}\, dx \right]$$

$$= \frac{1}{5}\left[0 - \frac{25}{n\pi}\cos(n\pi) + 0 - \left.\frac{\sin\left(\frac{n\pi x}{5}\right)}{(n\pi/5)^2}\right|_{0}^{5} \right]$$

$$= \frac{1}{5}\left[-\frac{25}{n\pi}(-1)^n - 0 \right] \qquad \{ \because \sin(n\pi) = 0 \}$$

$$\boxed{b_n = -\frac{5}{n\pi}(-1)^n}$$

Now fourier series from (1)...

$$f(x) = \frac{9}{4} + 5\sum_{n=1}^{\infty}\left[\frac{\left[(-1)^n - 1\right]}{n^2\pi^2}\cos\left(\frac{n\pi x}{5}\right) - \frac{\cancel{1}}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{5}\right) \right]$$

$$\therefore \boxed{f(x) = \frac{9}{4} + 5\sum_{n=1}^{\infty}\left\{ \frac{\left((-1)^n - 1\right)}{n^2\pi^2}\cos\left(\frac{n\pi x}{5}\right) - \frac{\cancel{1}}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{5}\right) \right\}}$$

Ans.

Note:
$\sin(n\pi) = 0$
$\cos(n\pi) = (-1)^n$