MAE 3360 Fall 2005
Engineering Analysis

Lecture Notes on:

Ordinary Differential Equations

Author: Professor Albert Y. Tong

Typist: Sakurako Takahashi

§ Linear Differential Equations of Higher Orders

As a first step, we must look at the idea of linear independence.

Consider the set of functions: $u_1(t)$, $u_2(t)$, $\cdots$, $u_n(t)$

A linear combination of these functions is:

$$\sum_{j=1}^{n} c_j u_j(t) \quad ; c_j = \text{constant}$$

If the equation:

$$\sum_{j=1}^{n} c_j u_j(t) = 0 \qquad t_1 < t < t_2$$

Can be satisfied with at least one $c_j \neq 0$ then the set of functions $u_j(t)$ are linearly dependent. The name arise because for $c_j \neq 0$, we can write

$$u_j(t) = \frac{1}{c_j}\left[c_1 u_1(t) + \cdots + c_{j-1} u_{j-1}(t) + c_{j+1} u_{j+1}(t) + \cdots + c_n u_n(t)\right]$$

Linear Independence occurs when $\sum_{j=1}^{n} c_j u_j(t) = 0$ is satisfied only by taking all $c_j = 0$.

Then $u_j(t)$ are linearly independent.

Usually, the state of the functions can be determined by inspection. In some cases, the following is useful:

Assume the $u_j(t)$ are linearly dependent. Then:

$$
\begin{aligned}
&c_1 u_1(t) + c_2 u_2(t) + \cdots + c_n u_n(t) = 0 \quad \text{(at least one } t\text{)}\\
&c_1 \frac{du_1}{dt} + c_2 \frac{du_2}{dt} + \cdots + c_n \frac{du_n}{dt} = 0\\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&c_1 \frac{d^{n-1}u_1}{dt^{n-1}} + c_2 \frac{d^{n-1}u_2}{dt^{n-1}} + \cdots + c_n \frac{d^{n-1}u_n}{dt^{n-1}} = 0
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix}
u_1 & u_2 & \cdots & u_n\\
\frac{du_1}{dt} & \frac{du_2}{dt} & \cdots & \frac{du_2}{dt}\\
\vdots & \vdots & & \vdots\\
\frac{d^{n-1}u_1}{dt^{n-1}} & \frac{d^{n-1}u_2}{dt^{n-1}} & \cdots & \frac{d^{n-1}u_n}{dt^{n-1}}
\end{bmatrix}
\begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ \vdots\\ 0 \end{bmatrix}
$$

Because at least one $c_j = 0$, the determinant of the matrix $(n \times n)$ denoted by $W(u_j)$ must be zero.

i.e. linear dependence $\Rightarrow W(u_j) = 0$

which gives: $W(u_j) \neq 0 \Rightarrow$ linear independence

Note: $W(u_j) = 0 \underbrace{\not\Rightarrow}_{\text{does not necessary imply}}$ linear dependence

Example

(1) $u_1 = \cos wx$, $u_2 = \sin wx$

$$
W(u_1, u_2) = \begin{vmatrix} \cos wx & \sin wx\\ -w\sin wx & w\cos wx \end{vmatrix} = w\left[\cos^2 wx + \sin^2 wx\right] = w \neq 0
$$

$\therefore u_1$ and u_2 are linearly independent.

(2) $u_1 = e^x$, $u_2 = xe^x$

$$
\begin{aligned}
W(u_1, u_2) &= \begin{vmatrix} e^x & xe^x\\ e^x & e^x(1+x) \end{vmatrix}\\
&= (e^x)(e^x)(1+x) - xe^{2x}\\
&= e^{2x}\left[1 + x - x\right] = e^{2x} \neq 0
\end{aligned}
$$

$\therefore u_1$ and u_2 are linearly independent.

(3) $u_1 = e^{mx}$, $u_2 = e^{nx}$ $\quad m \neq n$

$$W(u_1, u_2) = \begin{vmatrix} e^{mx} & e^{nx} \\ me^{mx} & ne^{nx} \end{vmatrix}$$
$$= ne^{(m+n)x} - me^{(m+n)x}$$
$$= (n-m)e^{(m+n)x} \neq 0$$

$\therefore e^{mx}$ and e^{nx} are linearly independent provided $m \neq n$

(4) Consider the set of polynomials

$$u_j(t) : 1 , t , t^2 , \cdots , t^n$$

$$W\big(u_j(t)\big) = \det \begin{vmatrix} 1 & t & t^2 & \cdots & \cdots & \cdots & t^n \\ 0 & 1 & 2t & \cdots & \cdots & \cdots & nt^{n-1} \\ 0 & 0 & 2 & \cdots & \cdots & \cdots & n(n-1)t^{n-2} \\ \vdots & \vdots & 0 & 3! & & & \vdots \\ \vdots & \vdots & \vdots & & 4! & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & n! \end{vmatrix}$$

$$\therefore W\big(u_j(t)\big) = (0!)(1!)(2!)\cdots(n!)$$
$$\neq 0$$

$\therefore u_j$'s are linearly independent.

<u>Existence and Representation Theorem</u>:

Given the ordinary, homogeneous, linear differential equation,

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + ya_n(t) = 0 \qquad (3)$$

in which $a_0(t)$, $a_1(t)$, … , $a_n(t)$ are continuous functions of t in some interval I of t , and which $a_0(t) \neq 0$.

(a) Equation (3) has n linearly independent solutions for t in I.

If $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ are n linearly independent solutions of Equation (3), then $y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ is also a solution, and conversely every solution of Equation (3) can be represented by an appropriate choice of the constant c_j.

(b) The inhomogeneous form of Equation (3) is given by

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n(t) y = f(t) \tag{4}$$

where $f(t)$ is continuous in I.

If $Y(t)$ is a solution to Equation (4), the

$$y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) \tag{5}$$

is a solution to Equation (4), and conversely, every solution of Equation (4) can be represented by an appropriate choice of the constants in Equation (5).

Existence and Uniqueness Theorem

Under the same condition as above, for every point t_0 in I, and for every set of constant k_j, $j = 1, 2, \cdots, n$, there is one and only one $y(t)$ to Equation (4) satisfying the condition

$$\begin{aligned} y(t_0) &= k_1 \\ y^1(t_0) &= k_2 \\ &\vdots \\ y^{n-1}(t_0) &= k_n \end{aligned}$$

(superscript denotes derivative)

§ Linear Differential Equations for Higher Orders

–Case of Constant Coefficient

For constant a_i, we have

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1} y' + a_n y = 0 \qquad (6)$$

(superscript denotes derivative).

Try $y = Ae^{mt}$

$y' = mAe^{mt}$, etc.

Equation (6) becomes:

$$m^n Ae^{mt} + a_1 m^{n-1} Ae^{mt} + \cdots + a_{n-1} mAe^{mt} + a_n Ae^{mt} = 0$$

$$\Rightarrow Ae^{mt}\left(m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n\right) = 0$$

If $A = 0$, then we have the trivial solution: $y = 0$ (which, in general, we don't want).

Therefore we have left the characteristic equation

$$\boxed{\left(m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n\right) = 0} \qquad (7)$$

In general, we have n-distinct solutions, m_j, $j = 1, 2, \ldots, n$. So we have for

$j = 1, 2, \ldots, n$, $y_j = A_j e^{m_j t}$

If this set is linearly independent (it is), the complete solution to the homogeneous equation is:

$$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + \cdots + A_n e^{m_n t}$$

Worked Example 1:

$$y'' - y = 0 \quad \leftarrow \text{homogeneous}$$

$$\Rightarrow m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$$\Rightarrow y_1 = e^t \text{ and } y_2 = e^{-t}$$

$$\Rightarrow \boxed{y = A_1 e^t + A_2 e^{-t}} \qquad (1)$$

This is called the general solution for the homogeneous differential equation. A_1 and A_2 are arbitrary constants. They will be determined by Initial or Boundary conditions.

e.g. Initial value provided

$$y'' - y = 0\,; \quad y(0) = 4\,,\; y'(0) = -2$$

(i) $y(0) = 0$:

$$0 = A_1 + A_2 = 4 \quad (2)$$

(ii) $y'(t) = A_1 e^t - A_2 e^{-t}$

$$\Rightarrow y'(0) = A_1 - A_2 = -2 \quad (3)$$

(2) and (3) $\Rightarrow A_1 = 1$; $A_2 = 3$

or $\boxed{y = e^t + 3e^{-t}}$

Worked Example 2:

$$y'' + y' - 2y = 0 \; ; \; y(0) = 4 \; , \; y'(0) = -5$$

$$\Rightarrow m^2 + m - 2 = (m-1)(m+2) = 0$$

$$\Rightarrow m = 1 \text{ and } m = -2$$

$$\Rightarrow \boxed{y = A_1 e^t + A_2 e^{-2t}} \quad (1)$$

Apply initial conditions

(i) $y(0) = 4$

$$\Rightarrow \boxed{4 = A_1 + A_2} \quad (2)$$

(ii) $y'(0) = -5$

$$y'(t) = A_1 e^t - 2A_2 e^{-2t}$$

$$\Rightarrow y'(0) = \boxed{A_1 - 2A_2 = -5} \quad (3)$$

(2) and (3) $\Rightarrow A_1 = 1$, $A_2 = 3$

$$\Rightarrow \boxed{y(t) = e^t + 3e^{-2t}}$$

In general form,

$$y'' + ay' + by = 0$$

$$\Rightarrow m^2 + am + b = 0$$

$$\Rightarrow m = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

3 cases:

(i) $a^2 - 4b > 0$: two real roots

(ii) $a^2 - 4b = 0$: one double root

(iii) $a^2 - 4b < 0$: two complex roots (conjugate pairs)

So far, we have taken case of (i).

The above outlines the typical case, but there are a number of features we must examine.

(i) Complex roots:

In some cases, the characteristic equation (Equation (7)) has roots that are complex, e.g. $m = a + ib$. To generate the real solution, we use Euler formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Note: If complex roots occur, they occur in conjugate pairs.

i.e. $m_1 = a + ib$
$m_2 = a - ib$

Example $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = 0$

let $y = e^{mt}$ gives:

$$\left(m^2 + 2m + 5\right)e^{mt} = 0$$
$$\Rightarrow m_{1,2} = \frac{-2 \pm \sqrt{4 - 4\times 5}}{2}$$
$$= -1 + 2i$$

so $y = A_1 e^{(-1+2i)t} + A_2 e^{(-1-2i)t}$

Note: A_1 and A_2 are complex for y to be real.

Let $A_1 = a_1 + ib_1$
$A_2 = a_2 + ib_2$

$$y = \left(a_1 + ib_1\right)e^{-t}e^{2it} + \left(a_2 + ib_2\right)e^{-t}e^{-i2t}$$
$$= e^{-t}\left[\left(a_1 + ib_1\right)\left(\cos 2t + i\sin 2t\right) + \left(a_2 + ib_2\right)\left(\cos 2t - i\sin 2t\right)\right]$$
$$= e^{-t}\left[\left(a_1 + a_2\right)\cos 2t - \left(b_1 - b_2\right)\sin 2t + i\left(b_1 + b_2\right)\cos 2t + i\left(a_1 - a_2\right)\sin 2t\right]$$

For y to be real, choose $a_1 = a_2$ and $b_1 = -b_2$

i.e. choose $\begin{matrix} A_1 = a_1 + ib_1 \\ A_2 = a_2 + ib_2 \end{matrix} \rightarrow$ Complex Conjugate

Take this to be the case, and calling $(a_1 + a_2)$, c_1 and $-(b_1 - b_2)$, c_2, we have as the solution

$$y = e^{-t}\left[c_1 \cos t + c_2 \sin 2t\right]$$

in this case c_1 and c_2 are real.

Worked example:

Recall, $y'' + y = 0$

$$\Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm\, i$$
$$\Rightarrow y = A_1 e^{it} + A_1 e^{-it}$$
$$= c_1 \cos t + c_2 \sin t$$

It has been shown that $\sin t$ and $\cos t$ are both solutions. Thus

$$y = c_1 \cos t + c_2 \sin t$$

is the general solution.

It is not necessary to do this every time. In the future, we should be able to see that we can go:

$$y = A_1 e^{-1+2it} + A_2 e^{-1-2it}$$
$$= e^{-t}\left[A_1 e^{2it} + A_2 e^{-2it}\right]$$

then we can say:

$$y = e^{-t}\left[c_1 \cos 2t + c_2 \sin 2t\right]$$

(ii) Repeated roots:

If the characteristic equation:

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n = 0$$

has a twice repeated root m_1

then one standard procedure will only generate $n-1$ linearly independent solution of the form

$$y_j(t) = A_j e^{m_j t}$$

Question: How do we generate the required n-th solution?

The two solutions corresponding to a repeated root are:

$$y_1 = A_1 e^{m_1 t}$$
$$y_2 = A_2 e^{m_1 t}$$

Note: This is valid only if the differential equation is linear and with constant coefficients.

Example: solve $\dfrac{d^3 y}{dt^3} - 5\dfrac{d^2 y}{dt^2} + 8\dfrac{dy}{dt} - 4y = 0$

characteristic equation: $m^3 - 5m^2 + 8m - 4 = 0$

$$\Rightarrow (m-1)(m-2)^2 = 0$$
$$\Rightarrow m_{1,2} = 2\,;\; m_3 = 1$$

$$y = A_1 e^{2t} + A_2 t e^{2t} + A_3 e^{t}$$

In the case of a root repeated p times, the solution of the general equation is:

$$y = A_1 e^{m_1 t} + A_2 t\, e^{m_1 t} + A_3 t^2 e^{m_1 t} + \cdots + A_p t^{p-1} e^{m_1 t} + A_{p+1} e^{m_{p+1} t} + A_{p+2} e^{m_{p+2} t} + \cdots + A_n e^{m_n t}$$

§ Equidimensional Equation

(Other names: Cauchy's Equation or Euler's Equation)

Consider:

$$x^n \frac{d^n y}{dx^n} + b_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_{n-1} x \frac{dy}{dx} + b_n y = 0$$

The equation can be reduced to equation with constant coefficient by a change in the independent variable:

Let $x = e^z$

$$dx = e^z dz$$

$$\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{e^z}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{1}{e^z}\frac{dy}{dz}\right)\frac{1}{e^z}$$

$$= \left(\frac{1}{e^z}\frac{d^2y}{dz^2} - \frac{1}{e^z}\frac{dy}{dz}\right)\frac{1}{e^z}$$

$$= \frac{1}{e^{2z}}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}\frac{d}{dz}\left(\frac{d}{dz} - 1\right)y$$

Similarly, it can be shown that:

$$\frac{d^m y}{dx^m} = \frac{1}{x^m}\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\cdots\left(\frac{d}{dz} - m + 1\right)y$$

Looking at terms in equation:

x - version		z - version
y	$\rightarrow$	y
$x\frac{dy}{dx}$	$\rightarrow$	$\frac{dy}{dz}$
$x^2\frac{d^2y}{dx^2}$	$\rightarrow$	$\frac{d^2y}{dz^2} - \frac{dy}{dz}$
$\vdots$	$\vdots$	$\vdots$
$x^m\frac{d^m y}{dx^m}$	$\rightarrow$	$\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\cdots\left(\frac{d}{dz} - m + 1\right)y$

With z the independent variable, the equation has constant coefficients and the solution is found from assuming $y = e^{mz} = x^m$. So we can solve the original homogeneous equation by assuming $y = x^m$.

Example:

Solve: $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$

(a) Try $y = x^m$

$$x^2 m(m-1)x^{m-2} + 2xmx^{m-1} - 2x^m = 0$$
$$\Rightarrow x^m\left(m(m-1)+2m-2\right)=0$$
$$\Rightarrow m^2+m-2=0$$
$$\Rightarrow (m+2)(m-1)=0$$
$$\Rightarrow m=-2\,;+1$$
$$\therefore y=\frac{A_1}{x^2}+A_2x$$

(b) Transform by $x=e^z$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0$$
$$\Rightarrow\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+2\frac{dy}{dz}-2y=0$$
$$\Rightarrow\frac{d^2y}{dz^2}+\frac{dy}{dz}-2y=0$$

Let $y=e^{mz}$ gives: $m^2+m-2=0$

$$\Rightarrow (m+2)(m-1)=0\,;\ m=-2\,;+1$$
$$\therefore y=c_1e^{-2z}+c_2e^z$$
$$=\frac{c_1}{x^2}+c_2x$$

Repeated roots:

In the z version, a repeated root would give:

$$y_1=A_1e^{m_1z}$$
$$y_2=A_2z\,e^{m_1z}$$

With $x=e^z$, $z=\ln x$

$$\therefore y_1=A_1x^{m_1}$$
$$y_2=A_2x^{m_1}\ln x$$

In the case of a root repeated p times, the corresponding part of the solution is

$$y=A_1x^{m_1}+A_2x^{m_1}\ln x+A_3x^{m_1}(\ln x)^2+\cdots+A_px^{m_1}(\ln x)^{p-1}$$

factor it out

$$y=x^{m_1}\left(A_1+A_2\ln x+A_3(\ln x)^2+\cdots+A_p(\ln x)^{p-1}\right)$$