## § Fourier Series

Periodic fcts occur frequently in engineering problems. Their representation in terms of simple periodic fcts, such as sine and cosine is a matter of great practical importance, which leads to Fourier series. These series are a very power tool in connection with various problems involving ordinary & partial differential eqns.

Periodic function: A fct $f(x)$ is said to be periodic if it is defined for all real $x$ and if there is some positive number $T$ s.t.

$$f(x+T) = f(x)$$

$T$: period of $f(x)$

e.g. $\sin x$ & $\cos x$

period $= 2\pi$

it follows that

$$f(x+nT) = f(x) \quad ; \quad n = \text{integer}$$

Let us assume that $f(x)$ is a periodic fct of period $2\pi$ which can be represented by a trigonometric series

$$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Given such a fct $f(x)$, we want to determine the coefficients $a_n$ & $b_n$

~~To deter~~

**$a_0$** :

Integrate (1) from $-\pi$ to $\pi$ :

$$(2) \quad \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx$$

If term-by-term integration of the series is allowed, then we obtain (again, it requires that the series is uniformly convergent)

$$\int_{-\pi}^{\pi} f(x)\,dx = \underbrace{a_0 \int_{-\pi}^{\pi} dx}_{= 2\pi a_0} + \sum_{n=1}^{\infty} \Big( a_n \underbrace{\int_{-\pi}^{\pi} \cos nx\,dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin nx\,dx}_{=0} \Big)$$

$$(3) \quad \therefore \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx}$$

**$a_n$** : $n = 1, 2, --$ (by a similar procedure):
multiply (1) by $\cos mx$, where $m$ is any "fixed" +ve integer, and then integrate from $-\pi$ to $\pi$ :

$$(4)\quad \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\Big[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\Big]\cos mx\,dx$$

Integrating term-by-term, r.h.s. becomes.

$$\underbrace{a_0\int_{-\pi}^{\pi}\cos mx\,dx}_{\parallel\;0} + \sum_{n=1}^{\infty}\Big[a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\Big]$$

Note: The first integral (with the $a_0$ term) is zero.

Also:

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\overset{0}{\overset{\parallel}{\cos(n+m)x}}\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\overset{0 \text{ except when } n=m\text{, and is equal to } 2\pi}{\overset{\parallel}{\cos(n-m)x}}\,dx$$

$$\&\quad \int_{-\pi}^{\pi}\sin nx\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\overset{0}{\overset{\parallel}{\sin(n+m)x}}\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\overset{0}{\overset{\parallel}{\sin(n-m)x}}\,dx$$

Integration shows that the four terms on the right are zero, except for the last ~~one~~ term in the first line which equals $\pi$ when $n=m$. and this term is multiply by $a_m$.

∴ ~~We have Eqn (5) ...~~

Eqn (4) becomes:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m\cdot\pi$$

$$(8) \quad \therefore \quad \boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \, dx}$$

$\underline{b_n}$ : ( similar procedure )
multiply (1) by $\sin mx$, ~~take~~ then integrate from $-\pi$ to $\pi$ ; we have

$$\int_{-\pi}^{\pi} f(x)\sin mx \, dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right]\sin mx \, dx$$

Integrate term-by-term, the r.h.s. becomes

$$\underbrace{a_0\int_{-\pi}^{\pi}\sin mx \, dx}_{= 0} + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx \sin mx \, dx + b_n\int_{-\pi}^{\pi}\sin nx \sin mx \, dx\right]$$

Note : $\displaystyle\int_{-\pi}^{\pi}\cos nx \sin mx \, dx = 0$ ( from the previous work )

Also

$$\int_{-\pi}^{\pi}\sin nx \sin mx \, dx = \underbrace{\frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x \, dx}_{=\begin{cases} 0 \, ; & n \neq m \\ 2\pi \, ; & n = m \end{cases}} - \underbrace{\frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x \, dx}_{= 0}$$

Again the non-zero term $(n=m)$ is multiplied by $b_m$ $\therefore$ the r.h.s. is equal to $b_m \pi$

$$\therefore \int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m \pi$$

$$\Longrightarrow (6) \quad \boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx}$$

<u>Summary</u> :

$$(7) \quad \begin{cases} f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$$

The series is called the ~~Fourier~~ <u>Fourier series</u> corresponding to $f(x)$, and its coefficients, $a_n$ & $b_n$'s are called the

Sometimes it is represented as: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$ ; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$

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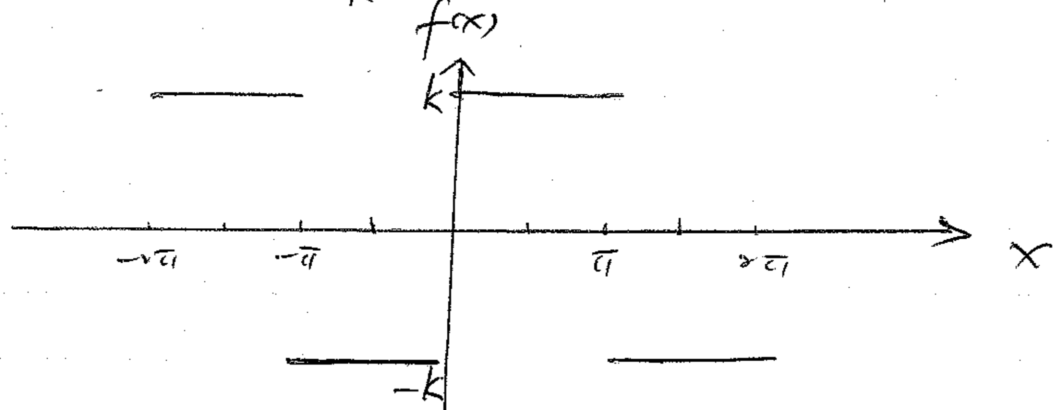
* Fourier coefficient of $f(x)$.

<u>Note</u>: Because of the periodicity of the integrands the interval of integration in (7) may be replaced by any other interval of length $2\pi$, for instance, by the interval $0 \le x \le 2\pi$.

<u>example</u>: <u>Square wave</u>

Find the Fourier coefficients of the periodic fct $f(x)$ given by:

$$f(x) = \begin{cases} -k & \text{when } -\pi < x < 0 \\ k & \text{"} \quad 0 < x < \pi \end{cases} \quad \& \quad f(x + 2n\pi) = f(x)$$




$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\Big\{\underbrace{\int_{-\pi}^{0} -k\, dx}_{-k\pi} + \underbrace{\int_{0}^{\pi} k\, dx}_{k\pi}\Big\} = 0$$

Note: $a_0$ is simply the average value of $f(x)$ over the period $2\pi$. In this case it is ~~one~~ obvious ~~that~~ the average value of $f(x)$ is zero.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

Since $\sin(\pm n\pi) = 0$, $a_n = 0$ for $n = 1, 2, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{k}{n\pi}\left[\underbrace{\cos 0}_{1} - \cos(-n\pi) - \cos n\pi + \underbrace{\cos 0}_{1}\right]$$

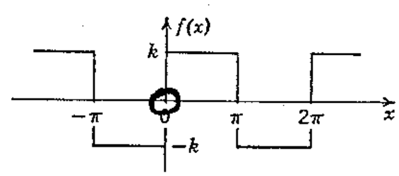
$$= \frac{2k}{n\pi}(1 - \cos n\pi)$$

$$\cos n\pi = \begin{cases} -1 & n = \text{odd} \\ 1 & n = \text{even} \end{cases}$$

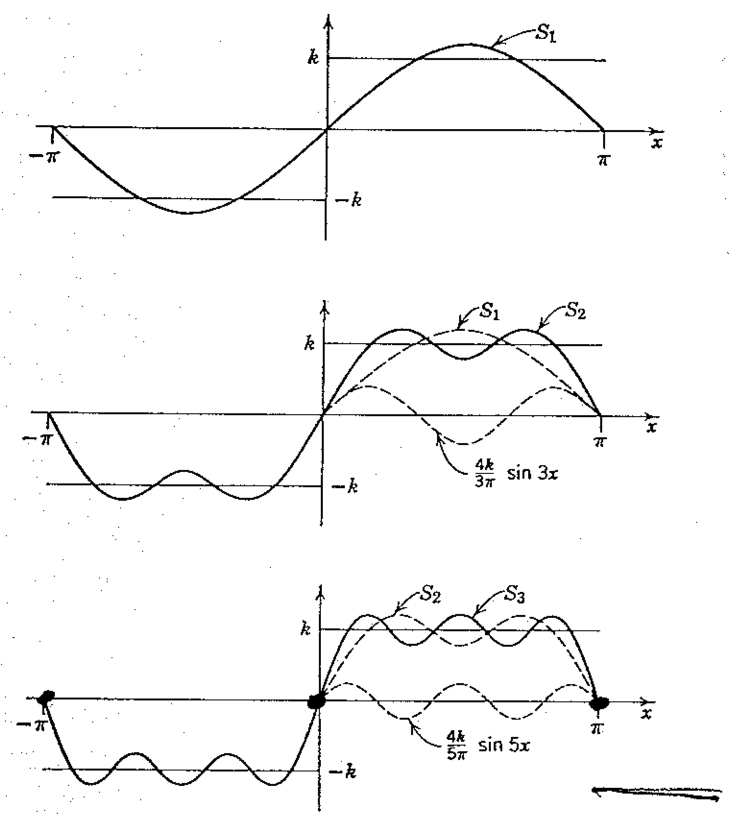
$$\therefore b_n = \cancel{\frac{2k}{n\pi}} \cdot \begin{cases} \dfrac{4k}{n\pi} & ; \ n = \text{odd} \\ 0 & ; \ n = \text{even} \end{cases}$$

$\therefore$ Fourier Series is

$$= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$




*(a)* The given function $f(x)$ (Periodic square wave)




*(b)* The first three partial sums of the corresponding Fourier series

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

it is a piecewise continuous and has a piecewise continuous derivative on the interval. A fct which is continuous and has a continuous derivative is called smooth.

<u>Convergence Theorem</u>: Let $f(x)$ be a piecewise-smooth function with period $2\pi$ and let

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad ; \; k = 1, 2, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \quad ; \quad k = 1, 2, \ldots$$

Then, for all $x$,

$$\frac{f(x^+) + f(x^-)}{2} = a_0 + \sum_{k=1}^{\infty}(a_k\cos kx + b_k \sin kx)$$

§ <u>Functions having arbitrary period</u>

The transition from fcts having period $2\pi$ to functions having period $T$ can be effected by a change of scale. (stretching)

Suppose $f(t)$ has period $T$.
Introduce a new variable $x$ such that $f(t)$, as a fct of $x$, has period $2\pi$. We set,

(a) $t = \frac{T}{2\pi}x$ so that (b) $x = \frac{2\pi}{T}t$

then $x = \pm\pi$ corresponds to $t = \pm T/2$

This means ~~that~~ $f$, as a fct of $x$ has period $2\pi$. Hence if $f$ has a Fourier Series, this series must be of the form

$$f(t) = f\left(\frac{T}{2\pi}x\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with coefficients obtained from:

(i) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) dx$$

(ii) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) \cos nx \, dx$$

(iii) $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) \sin nx \, dx$$

We could use these formulas directly, but the change to $t$ simplifies calculation.

$$x = \frac{2\pi}{T}t \Rightarrow dx = \frac{2\pi}{T}dt \quad ; \quad x = -\pi \Rightarrow t = \frac{-T}{2} \quad ; \quad x = +\pi \Rightarrow t = \frac{+T}{2}$$

(i) becomes:

$$a_0 = \frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\left(\frac{2\pi}{T}dt\right) = \boxed{\frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\, dt}$$

(ii) becomes

$$a_n = \frac{1}{\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n \frac{2\pi t}{T} \, dt \, \frac{2\pi}{T}; \quad n = 1, 2, \ldots$$

$$\boxed{a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{T/2} f(t) \cos \frac{2n\pi t}{T} \, dt \; ; \quad n = 1, 2, \ldots}$$

Similarly, (iii) becomes

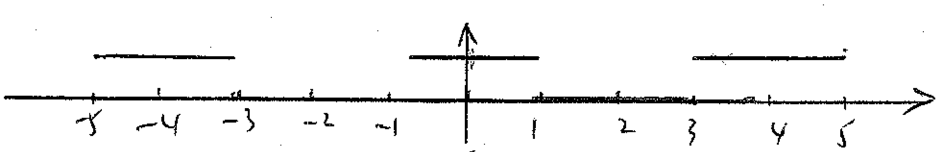
$$\boxed{b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{T/2} f(t) \sin \frac{2n\pi t}{T} \, dt \; ; \quad n = 1, 2, \ldots}$$

Furthermore, the Fourier series with $x$ expressed in terms of $t$ becomes

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi}{T} t + b_n \sin \frac{2n\pi}{T} t \right)}$$

Example: Periodic square wave

$$f(t) = \begin{cases} 0 & \text{when } -2 < t < -1 \\ k & \text{"} \quad -1 < t < 1 \\ 0 & \text{"} \quad 1 < t < 2 \end{cases} \quad \Bigg\} \; T = 4 \text{ (period)}$$



$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt = \frac{1}{4}\int_{-2}^{2} f(t)\,dt$$

$$= \frac{1}{4}\int_{-1}^{1} k\,dt = \frac{k}{2}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\frac{2n\pi t}{T}\,dt$$

$$= \frac{2}{4}\int_{-2}^{2} f(t)\cos\frac{2n\pi t}{4}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi t}{2}\,dt$$

$$= \frac{k}{2}\cdot\frac{2}{n\pi}\sin\frac{n\pi t}{2}\Big|_{-1}^{1}$$

$$= \frac{k}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right)$$

$$= \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Note: $a_n = 0$ when $n$ is even

~~& $a_n = \frac{2k}{n\pi}$ is $n$ is odd.~~

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\frac{2n\pi t}{T}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} k \sin\frac{n\pi t}{2}\, dt = 0$$

Hence the result is

$$\boxed{f(t) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \cdots + \cdots\right)}$$

§ <u>Even & Odd Functions</u>

Unnecessary work in determining Fourier Coefficients of a fct can be avoided if the fct is odd or even.

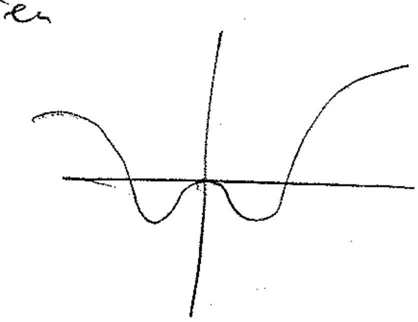
A fct $y = g(x)$ is said to be even if: $g(-x) = g(x)$ for all $x$.

~~A fct~~ is said to be odd if $g(-x) = -g(x)$ for all $x$.

The fct $\cos nx$ is even, while the $\sin nx$ is odd.

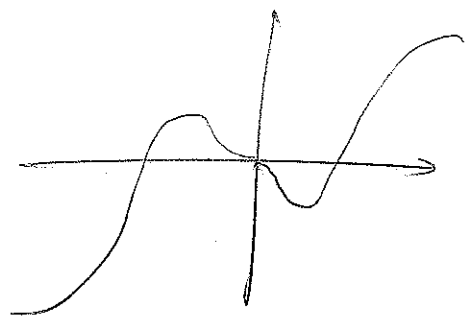
If $g(x)$ is an <u>even</u> fct, then

$$\int_{-T/2}^{T/2} g(x)\,dx = 2\int_{0}^{T/2} g(x)\,dx$$



If $h(x)$ is an <u>odd</u> fct, then

$$\int_{-T/2}^{T/2} h(x)\,dx = 0$$



The product $\mathfrak{g} = gh$ of an even fct $g$ and an odd fct $h$ is odd, because

$$\mathfrak{g}(-x) = g(-x)h(-x) = g(x)(-h(x))$$
$$= -g(x)h(x) = -\mathfrak{g}(x)$$

Hence if $f(t)$ is even, then the integrand $f(t)\sin\left(\frac{2n\pi t}{T}\right)$ in the integral for ~~determining~~ $b_n$ is odd, subsequently $b_n = 0$.

Similarly, if $f(t)$ is odd, then $f(t)\cos\frac{2n\pi t}{T}$ is odd, and $a_n = 0$.

<u>Theorem</u>: <u>Fourier series of even and odd fcts</u>

(i) The Fourier series of an even fct $f(t)$ of period $T$ is a "<u>Fourier Cosine Series</u>"

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T}t$$

with coefficients

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt \;;$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2n\pi}{T}t\,dt \quad ; n=1,2,\ldots$$

(ii) The Fourier series of an odd fct $f(t)$ of period $T$ is a "<u>Fourier Sine series</u>"

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi}{T}t$$

with coefficients

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2n\pi}{T}t\,dt \quad ; \quad n=1,2,\dots$$

In particular, the Fourier series of an even fct $f(x)$ of period $2\pi$ is given by

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + \dots$$

with

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$n=1,2,\dots$$

Similarly, the Fourier series of an odd fct $f(x)$ of period $2\pi$ is a Fourier sine series

$$f(x) = b_1\sin x + b_2\sin 2x + \dots$$

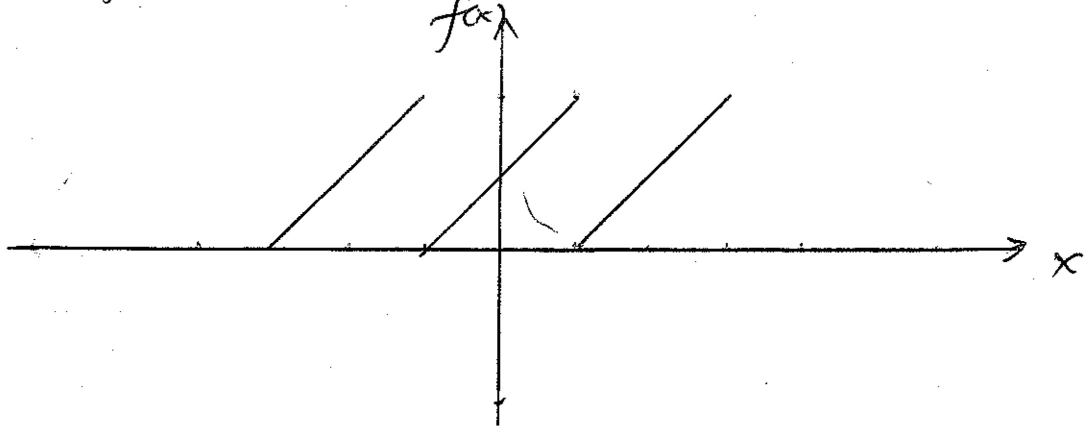
with

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad ; \quad n=1,2,\dots$$

~~Theorem Sum of~~

Example : ( Saw-toothed wave )

$f(x) = x + \pi$ when $-\pi < x < \pi$

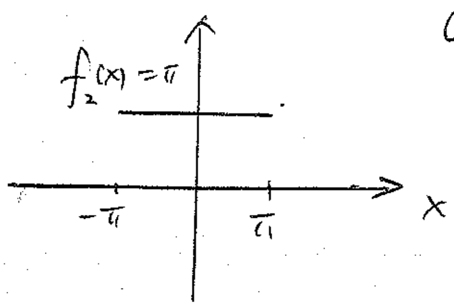


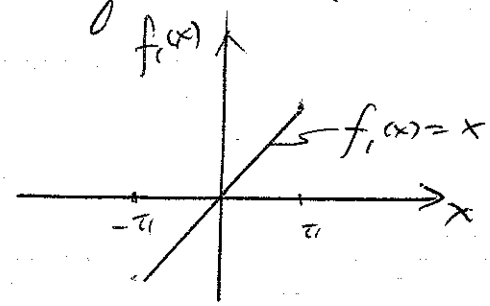

Note: The Fourier coeff of a sum $f_1 + f_2$ are the sums of corresponding Fourier coefficients of $f_1$ & $f_2$.

let $f_1 = x$ & $f_2 = \pi$

& $f(x) = f_1 + f_2$

(Note: basically we shift the level by $\pi$)



$\underline{f_2}$ : The only nonzero Fourier coefficient is $a_0$ (the constant term), which is $\pi$. (the average value of $f_2$)

$\underline{f_1}$ : $f_1$ is an odd fct, so $a_n = 0$ for all $n$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[ \frac{-x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx \right]$$

$$= \frac{2}{\pi}\left[ -\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\overset{0}{\cancel{\sin nx}}\Big|_0^{\pi} \right]$$

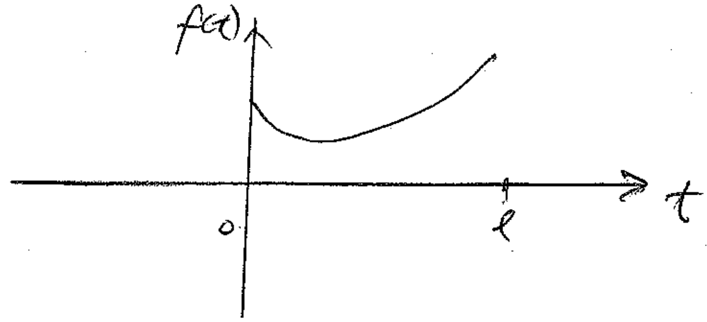
$$= -\frac{2}{n}\cos n\pi$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\therefore f(x) = \pi + 2\left( \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots \right)$$

§ Half-range expansions (odd + even periodic extensions)

• Suppose $f(t)$ is defined on an interval $0 \le t \le l$ and on this interval we want to represent $f(t)$ by a Fourier series.




There are two ways to do it:

(i) even periodic extension

we let the interval $0 \le t \le l$ correspond to the interval of integration $0 \le t \le T/2$, that is we set $T/2 = l$ or $T = 2l$, we then obtain a Fourier cosine series which represents an even periodic fct $f_1(t)$ of period $T = 2l$.

By construction $f_1(t) = f(t)$ on $0 \le t \le l$.

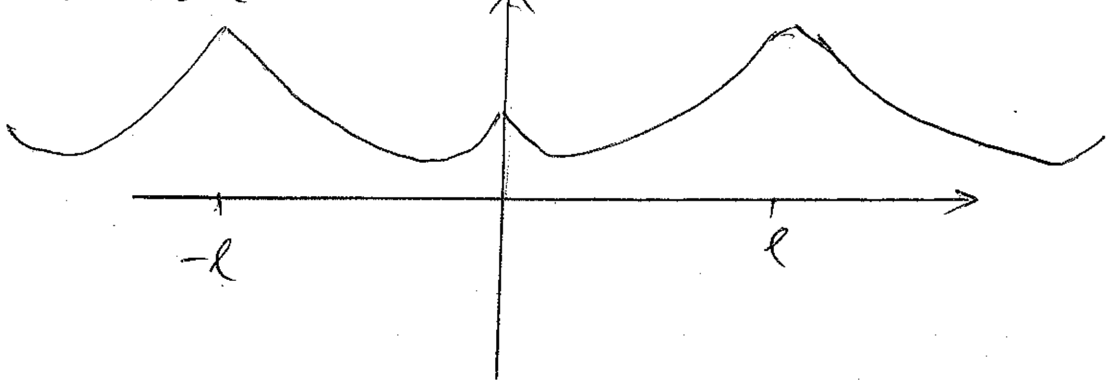
+ $f_1(t)$ is called the even periodic extension of $f(t)$ of period $2l$.

+ $$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} t \qquad 0 \le t \le l$$

$$a_0 = \frac{1}{l}\int_0^l f(t)\,dt \quad ; \quad a_n = \frac{2}{l}\int_0^l f(t)\cos\frac{n\pi t}{l}\,dt$$

$n = 1, 2, \ldots$

we have




(ii) odd periodic extension

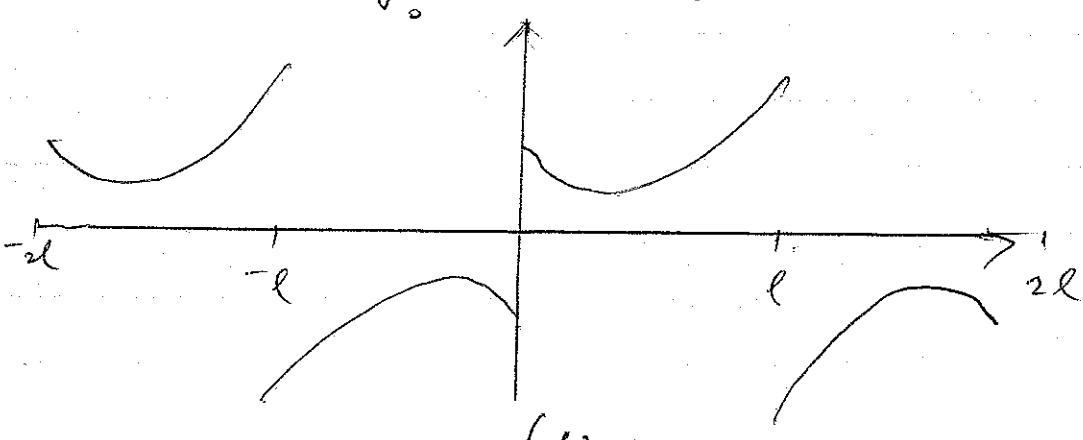
Instead of cosine Fourier series, we can obtain a Fourier sine series, which represents an odd periodic fct, say, $f_2(t)$, of period $T=2\ell$

By construction $f_2(t) = f(t)$ on $0 \le t \le \ell$

& $f_2(t)$ is called the odd periodic extension of $f(t)$ of period $2\ell$. we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{\ell}$$

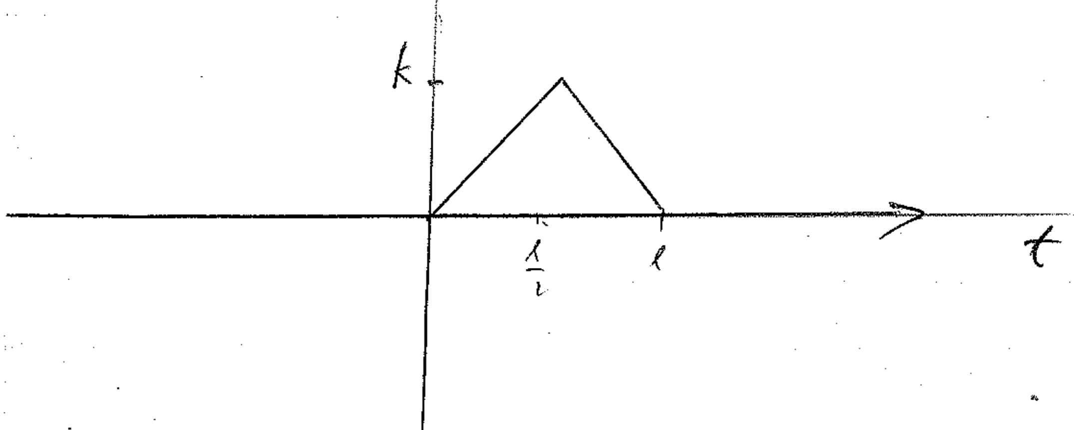
& $$b_n = \frac{2}{\ell} \int_0^{\ell} f(t) \sin \frac{n\pi t}{\ell} \, dt$$



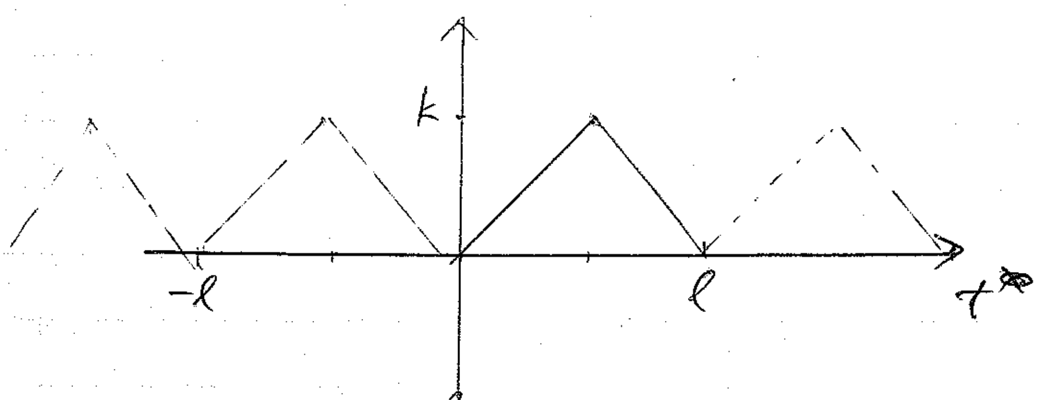
Example : (Triangular pulse)

Find the half-range expansions of the fct

$$f(t) = \begin{cases} \frac{2k}{l} t & \text{when } 0 < t < \frac{l}{2} \\ \frac{2k}{l}(l-t) & \text{when } \frac{l}{2} < t < l \end{cases}$$




(i) even extension




$$a_0 = \frac{1}{l}\int_0^l f(t)\,dt = \frac{1}{l}\left[\frac{2k}{l}\int_0^{l/2} t\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\,dt\right]$$

$$= \frac{k}{2}$$

$$a_n = \frac{2}{l}\left[\frac{2k}{l}\int_0^{l/2} t\cos\frac{n\pi t}{l}\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\cos\frac{n\pi t}{l}\,dt\right]$$

$$\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt = \frac{lt}{n\pi}\sin\frac{n\pi t}{l}\Big|_0^{l/2} - \frac{l}{n\pi}\int_0^{l/2}\sin\frac{n\pi}{l}t\,dt$$

$$= \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

Similarly

$$\int_{l/2}^{l}(l-t)\cos\frac{n\pi t}{l}\,dt = \frac{-l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

Substituting into $a_n$:

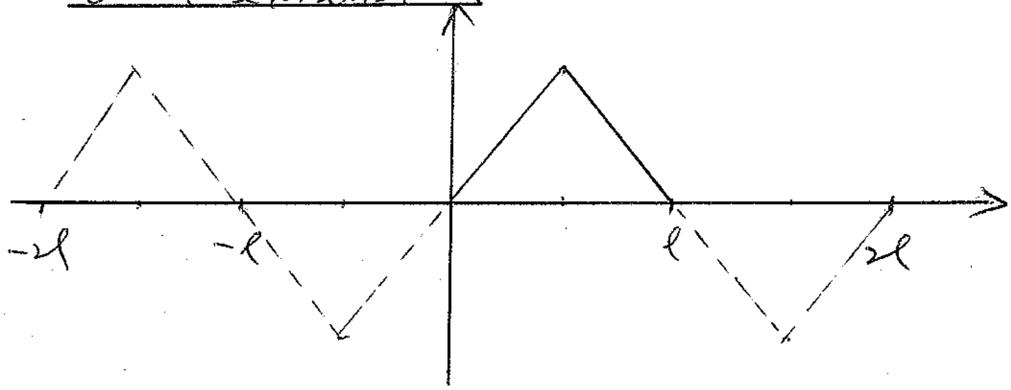
$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

$$a_1 = 0\ ;\ a_2 = -\frac{16k}{2^2\pi^2}\ ;\ a_3 = 0\ ;\ a_4 = 0,\ a_5 = 0$$

$$a_6 = \frac{-16k}{6^2\pi^2}\ ;\ a_7 = a_8 = a_9 = 0\ ;\ a_{10} = \frac{-16k}{10^2\pi^2}$$

$$f(t) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{l}t + \frac{1}{6^2}\cos\frac{6\pi}{l}t + \cdots\right)$$

(ii) odd extension




$$b_n = \frac{2}{l}\left[\frac{2k}{l}\int_0^{l/2} t \sin\frac{n\pi t}{l}\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\sin\frac{n\pi t}{l}\,dt\right]$$

after some algebra:

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(t) = \frac{8k}{\pi^2}\left(\sin\frac{\pi t}{l} - \frac{1}{3^2}\sin\frac{3\pi}{l}t + \frac{1}{5^2}\sin\frac{5\pi}{l}t + \dots\right)$$

9/8/03

§ Worked example

Expand $f(x) = x^2$, $0 \le x < 2\pi$ in a Fourier series of period $2\pi$

$$a_n = \frac{1}{\ell}\int_c^{c+2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}$$

$$= \frac{4}{n^2} \qquad n \neq 0$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}$$

$$= -\frac{4\pi}{n}$$

$$\therefore f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

@ $x=0$, $\dfrac{f(0^+)+f(0^-)}{2} = \dfrac{1}{2}(0 + 4\pi^2) = 2\pi^2$

then $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2} = 2\pi^2 \implies \boxed{\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$