

MAE 3360 Engineering Analysis II
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Lecture Note on Partial Fractions

Simple Rules on Partial Fractions

- i. If the degree of the numerator of the given function is equal to or greater than that of the denominator, divide the numerator by the denominator until a remainder is obtained which is of lower degree than the denominator.

e.g.
$$\frac{x^3}{x^2 - 3x + 2} = x + 3 + \frac{7x - 6}{x^2 - 3x + 2}$$

- ii. To every linear factor like $(x - a)$ in the denominator, there corresponds a partial fraction of the form $A/(x - a)$.

e.g.
$$\frac{5}{x^2 + x - 6} = \frac{5}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$$

- iii. To every repeated linear factor like $(x - a)^2$ in the denominator, there corresponds two partial fractions of the form $A/(x - a)$ and $B/(x - a)^2$. Similarly for factors like $(x - a)^3$ we have three partial fractions $A/(x - a)$, $B/(x - a)^2$ and $C/(x - a)^3$, and so on.

e.g.
$$\frac{9}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

- iv. To every quadratic factor like $x^2 + ax + b$, there corresponds a partial fraction $(Cx + D)/(x^2 + ax + b)$. Repeated quadratic factors require additional partial fractions as in (iii) above. Thus a factor $(x^2 + ax + b)^2$ would require partial fractions $(Cx + D)/(x^2 + ax + b)$ and $(Ex + F)/(x^2 + ax + b)^2$.

e.g.
$$\frac{16x}{x^4 - 16} = \frac{16x}{(x-2)(x+2)(x^2+4)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{Cx+D}{(x^2+4)}$$

Note: it is assumed that the quadratic factor has no real linear factors. Otherwise, it can be factored into $(x - \alpha)(x - \beta)$ if α & β are the real roots of the quadratic. In this case, rule (ii) will apply.

Examples on partial fractions expansion

$$1) \frac{3x^3 - x^2 + 2}{x(x^2 - 1)} = 3 + \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$2) \frac{2}{x^4 + x^2 + 1} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1}$$

Note that $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$

$$3) \frac{5x - 12}{(x + 2)(x^2 - 2x + 3)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 3}$$

Note that $x^2 - 2x + 3$ is an irreducible quadratic factor, i.e. it has no real linear factors.

$$4) \frac{2x^2 + x - 2}{x^3(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 1}$$

$$5) \frac{x^3 + 5x^2 + 4x + 5}{(x - 1)^2(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + x + 1}$$

$$6) \frac{2x^2 + x + 4}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{B_1x + C_1}{x^2 + 2} + \frac{B_2x + C_2}{(x^2 + 2)^2}$$

Techniques of obtaining the coefficients in partial fractions

(i) Non-repeated linear factors

Consider $f(x) = \frac{cx + d}{(x - a)(x - b)}$ where $a \neq b$

Let us assume that (see simple rule (ii)):

$$\frac{cx + d}{(x - a)(x - b)} = \frac{A}{(x - a)} + \frac{B}{(x - b)} \quad (1)$$

Multiplying both sides by $(x - a)(x - b)$, we have:

$$cx + d = A(x - b) + B(x - a) \text{ and equating coefficients:}$$

$$A + B = c \quad \text{and} \quad Ab + Ba = -d$$

Thus the unknown constants A and B are determined by these two equations.

In practice, it is better not to obtain A and B by solving the equations. Instead, we multiply Equation (1) by $(x - a)$ obtaining:

$$\frac{cx + d}{(x - b)} = A + \frac{B(x - a)}{(x - b)}$$

Putting $x = a$, we have

$$A = \frac{ca + d}{(a - b)}$$

Similarly, multiplying Equation (1) by $(x - b)$ and then putting $x = b$, we have

$$B = \frac{cb + d}{(b - a)}$$

In practice, when experience has been gained, it is not necessary to write down all the steps; for A and B may be obtained by inspection. When Equation (1) is multiplied by $(x - a)$ and x is then put equal to a, the right hand side reduces to A, and the left hand side is the value of $(cx + d)/(x - b)$ evaluated at $x = a$, and this is $(ca + d)/(a - b)$. All this can be done mentally. Similarly, B may be found.

Example:

$$\frac{x - 1}{(x + 1)(x + 3)} = \frac{A}{(x + 1)} + \frac{B}{(x + 3)}$$

$$A = \left. \frac{x-1}{(x+3)} \right|_{x=-1} = -1$$

$$B = \left. \frac{x-1}{(x+1)} \right|_{x=-3} = 2$$

$$\therefore \frac{x-1}{(x+1)(x+3)} = -\frac{1}{(x+1)} + \frac{2}{(x+3)}$$

Note: When doing numerical work, the student should make a habit of checking the arithmetic. This may be done by inserting suitable values for x and seeing whether they satisfy the expansion which has been established. For example, if we put $x=1$ we have $0 = -1/2 + 2/4$ which is correct.

(ii) Repeated linear factors

Consider $f(x) = (px^2 + qx + r)/(x-a)(x-b)^2$

From simple rule (iii), we have

$$f(x) = \frac{(px^2 + qx + r)}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{D}{(x-b)^2}$$

The coefficients A and D can be obtained using the technique just outlined in (i) for the non-repeated factor. Thus A is the value of $(px^2 + qx + r)/(x-b)^2$ evaluated at $x = b$ and D is that of $(px^2 + qx + r)/(x-a)$ evaluated at $x = b$. B can then be obtained afterward by inserting any convenient value for x in the expansion.

$$\text{e.g. } f(x) = \frac{1}{(x-1)(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{D}{(x-2)^2}$$

$$A = \left. \frac{1}{(x-2)^2} \right|_{x=1} = 1$$

$$D = \left. \frac{1}{(x-1)} \right|_{x=2} = 1$$

We have:

$$f(x) = \frac{1}{(x-1)(x-2)^2} = \frac{1}{(x-1)} + \frac{B}{(x-2)} + \frac{1}{(x-2)^2}$$

Let $x = 0$:

$$f(0) = \frac{1}{(0-1)(0-2)^2} = \frac{1}{(0-1)} + \frac{B}{(0-2)} + \frac{1}{(0-2)^2} \rightarrow B = -1.$$

A more general method capable of handling higher order repeated factors like $(x-a)^3$ is given in the ME5332 lecture notes.

(iii) Irreducible quadratic factor

Again, a more advanced treatment is given in the ME5332 lecture notes. Here a simple approach is discussed. Consider:

$$f(x) = \frac{x}{(x-3)(x^2+x+1)}$$

Here, the expansion is given by

$$f(x) = \frac{A}{(x-3)} + \frac{Bx+C}{(x^2+x+1)}$$

By the usual process, we find that A is the value of $x / (x^2+x+1)$ evaluated at $x=3$ which gives $A = 3/13$, and we have:

$$\begin{aligned} \frac{Bx+C}{x^2+x+1} &= \frac{x}{(x-3)(x^2+x+1)} - \frac{3}{13(x-3)} = \frac{13x-3x^2-3x-3}{13(x-3)(x^2+x+1)} \\ &= \frac{-(3x-1)(x-3)}{13(x^2+x+1)(x-3)} = \frac{-(3x-1)}{13(x^2+x+1)} \end{aligned}$$

Hence, the given expansion is equal to:

$$\frac{3}{13(x-3)} - \frac{3x-1}{13(x^2+x+1)}$$

Alternatively, B and C can be found by inserting two convenient values for x and solving for the resulting equations for B and C. The method shown is generally shorter and has the advantage of being self-checking. Note that (x-3) must be a factor of the numerator of the fraction formed; otherwise, an error must have occurred.