MAE3360 Engineering Analysis II
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Lecture Notes on Variation of parameters

(The method is due to Lagrange. It is applicable to linear differential equations with variable coefficients.)

Consider :

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$

We assume that the homogeneous solution is available and is given by $y_1(x)$ and $y_2(x)$, i.e.:

$$y_h = c_1 y_1(x) + c_2 y_2(x) \quad (2)$$

For the nonhomogeneous (i.e. particular) solution, assume:

$$y_p = u(x)y_1(x) + v(x)y_2(x) \quad (3)$$

Differentiating both sides gives:

$$\begin{aligned} y_p' &= u'y_1 + uy_1' + v'y_2 + vy_2' \\ &= (u'y_1 + v'y_2) + (uy_1' + vy_2') \end{aligned} \quad (4)$$

Set

$$u'y_1 + v'y_2 = 0 \quad (5)$$

leaving

$$y_p' = uy_1' + vy_2' \quad (6)$$

Then, differentiating gives:

$$y_p'' = (uy_1'' + vy_2'') + (u'y_1' + v'y_2') \quad (7)$$

Substituting Equations (3, 6 & 7) into Equation (1) gives:

$$(uy_1'' + vy_2'') + (u'y_1' + v'y_2') + p(x)(uy_1' + vy_2') + q(x)(uy_1 + vy_2) = r(x) \quad (8)$$

Collecting terms containing u and v yields;

$$u\left[y_1'' + p(x)y_1' + q(x)y_1\right] + v\left[y_2'' + p(x)y_2' + q(x)y_2\right] + (u'y_1' + v'y_2') = r(x) \quad (9)$$

Since y_1 & y_2 are the homogenous solutions, the first two terms are zero. Hence:

$$(u'y_1' + v'y_2') = r(x) \quad (10)$$

Equation (10) along with Equation (5) form the system of for $u' \& v'$:

$$(u'y_1 + v'y_2) = 0$$

$$(u'y_1' + v'y_2') = r(x)$$

$$\Rightarrow u'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ r(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -\frac{r(x)y_2(x)}{W\left(y_1(x), y_2(x)\right)} \qquad (11)$$

and

$$v'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & r(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{r(x)y_1(x)}{W\left(y_1(x), y_2(x)\right)} \qquad (12)$$

Note that, since $y_1 \& y_2$ are linearly independent, $W\left(y_1, y_2\right) \neq 0$. Therefore, a unique particular solution exists for $u' \& v'$. After solving $u' \& v'$, as given in Equations 11 and 12, they can be integrated to get $u \& v$, i.e.

$$\boxed{u(x) = -\int \frac{r(x)y_2(x)}{W\left(y_1(x), y_2(x)\right)}dx} \qquad \boxed{v(x) = \int \frac{r(x)y_1(x)}{W\left(y_1(x), y_2(x)\right)}dx} \qquad (13)$$

Example: *Find the general solution of the following D.E.*

$y'' + y = \sec(x)$

Solution:

Homogenous solution: $\lambda^2 + 1 = 0 \rightarrow y_h = Ay_1 + By_2$, A & B are constants,

where $\begin{cases} y_1(x) = \cos(x) \\ y_2(x) = \sin(x) \end{cases} \Rightarrow \begin{cases} y_1'(x) = -\sin(x) \\ y_2'(x) = \cos(x) \end{cases}$

The particular solution is given by:

$$\begin{aligned} y_p &= u(x)y_1(x) + v(x)y_2(x) \\ &= u(x)\cos(x) + v(x)\sin(x) \end{aligned}$$

From $\begin{array}{l}(u'y_1 + v'y_2) = 0 \\ (u'y_1' + v'y_2') = r(x)\end{array}$

They become: $\begin{array}{l}u'\cos(x) + v'\sin(x) = 0 \\ u'(-\sin(x)) + v'\cos(x) = \sec(x)\end{array}$ (1)

$$\Rightarrow u'(x) = \frac{\begin{vmatrix} 0 & \sin(x) \\ \sec(x) & \cos(x) \end{vmatrix}}{\begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}} = -\sec(x)\sin(x) = -\tan(x)$$

and

$$v'(x) = \frac{\begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec(x) \end{vmatrix}}{\begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}} = \sec(x)\cos(x) = 1$$

Therefore,

$$u(x) = \int u'dx = \int -\tan(x)dx \rightarrow \boxed{u(x) = \ln|\cos(x)|}$$

and $v(x) = \int v'dx = \int 1dx \rightarrow \boxed{v(x) = x}$

The particular solution is given by:

$$y_p = \ln|\cos(x)|\cos(x) + x\sin(x)$$

The general solution of the D.E is the sum of the homogenous and the nonhomogenous solutions, i.e.

$$y = y_h + y_p$$

$$\boxed{y = A\cos(x) + B\sin(x) + \ln|\cos(x)|\cos(x) + x\sin(x)}$$

Remark:

1-If integrating constants are included in the u' & v', i.e.

$$u(x) = \int u'dx + A = \int -\tan(x)dx + A \rightarrow \boxed{u(x) = \ln|\cos(x)| + A}$$

$$v(x) = \int v' dx + B = \int 1 dx + B \rightarrow \boxed{v(x) = x + B}$$

Then:

$$\begin{aligned} y &= \left[\ln\left|\cos(x)\right| + A \right]\cos(x) + \left[x + B\right]\sin(x) \\ &= \cos(x)\ln\left|\cos(x)\right| + x\sin(x) + A\cos(x) + B\sin(x) \end{aligned}$$

which is the same as the general solution.
