MAE 3360 Engineering Analysis

Lecture Notes (Part I)

Vector Calculus

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§ Differentiation of Vectors

Continuity : A vector function $\vec{u}(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighbourhood of t_0 and

$$\lim_{t \to t_0} \vec{u}(t) = \vec{u}(t_0)$$

If we introduce a Cartesian coordinate system,

$$\vec{u}(t) = u_1(t)\, i + u_2(t)\, j + u_3(t)\, k$$

then $\vec{u}(t)$ is continuous at t_0 if and only iff its three components (u_1, u_2, u_3) are continuous at t_0

Differentiability : A vector function $\vec{u}(t)$ is said to be differentiable at a pt t if the limit

$$\vec{u}'(t) = \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}$$

exists. The vector $\vec{u}'(t)$ is called the derivative of $\vec{u}(t)$

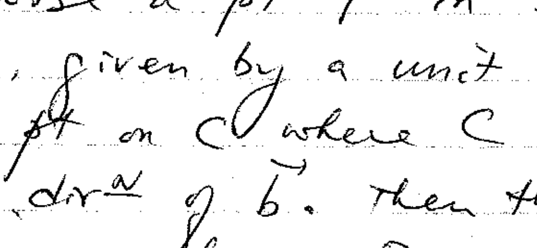
Note : To differentiate a vector fct, one differentiates each component separately. It can be shown that :

$$(i)\ (c\vec{u})' = c\,\vec{u}' \quad (c = \text{constant})$$
$$(ii)\ (\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$$
$$(iii)\ (\vec{u}\cdot\vec{v})' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$
$$(iv)\ (\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$$
$$(v)\ (\vec{u}\cdot\vec{v}\times\vec{w})' = \vec{u}'\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\vec{v}'\times\vec{w} + \vec{u}\cdot\vec{v}\times\vec{w}'$$

§ The Gradient Vector

We consider a scalar field given by a scalar function $f(x,y,z)$. We know that the first partial derivatives of f, i.e. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ & $\frac{\partial f}{\partial z}$ are the rates of change of f in the x, y, & z directions, respectively. We may ask for the rate of change of f in any direction. This leads to the notion of directional derivatives.

Consider the following diagram:



We choose a pt P in space and a dir$^{\underline{n}}$ at P, given by a unit vector $\vec{b}$. Q can be a pt on C where C is the ray from P in the dir$^{\underline{n}}$ of $\vec{b}$. Then the directional derivative of f at P in the direction of $\vec{b}$

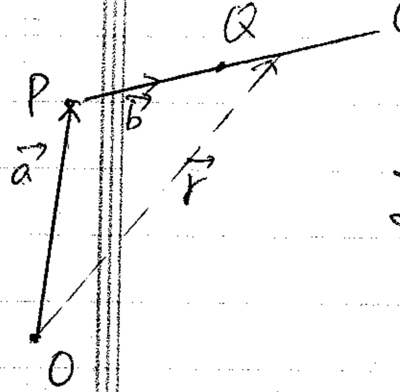
is given by:

$$\frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \qquad ①$$

where s = distance between P & Q. Obviously $\frac{df}{ds}$ is the rate of change of f at P in the direction of $\vec{b}$.

If P has the position vector $\vec{a}$, then the ray C can be represented in the form

$$\vec{r}(s) = x(s)\, i + y(s)\, j + z(s)\, k \qquad ②$$
$$= \vec{a} + s\,\vec{b}$$



$$\frac{df}{ds} = \frac{d}{ds}\left(f(x(s), y(s), z(s))\right)$$
$$= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \qquad ③$$

From ②:

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\, i + \frac{dy}{ds}\, j + \frac{dz}{ds}\, k$$
$$= \frac{d\vec{a}}{ds} + \frac{d}{ds}(s\vec{b}) = \vec{b}$$

This suggests that if we introduce the vector

$$\text{grad } f = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k \qquad ④$$

Eq (3) can be written as:

$$\frac{df}{ds} = \nabla f \cdot \vec{b} \qquad (5)$$

The vector grad f is called the gradient of the scalar function f.

By introducing the differential operator $\nabla \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$, we have

$$\text{grad } f \equiv \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

e.g. $\vec{b} = \vec{i}$ (i.e. directional derivative in the x-dir^{n})

$$\frac{\partial f}{\partial s} = \vec{b} \cdot \nabla f = \vec{i} \cdot \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$= \frac{\partial f}{\partial x} \quad \text{(as expected)}$$

Similarly for $\vec{b} = \vec{j}, \vec{k}$ for $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$.

<u>Example</u>: Find the directional derivative $\frac{\partial f}{\partial s}$ of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $(2,1,3)$ in the direction of the vector $\vec{a} = \vec{i} - 2\vec{k}$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\nabla f = 4x\,\vec{i} + 6y\,\vec{j} + 2z\,\vec{k}$$

and at $P(2, 1, 3)$,

$$\nabla f = 8\vec{i} + 6\vec{j} + 6\vec{k}$$

$$\vec{a} = \vec{i} - 2\vec{k}$$

$\vec{b}$ = unit vector in the direction of $\vec{a}$

$$= \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} - 2\vec{k}}{\sqrt{5}}$$

$$\therefore \frac{\partial f}{\partial s} = (8\vec{i} + 6\vec{j} + 6\vec{k}) \cdot \left(\frac{\vec{i} - 2\vec{k}}{\sqrt{5}}\right)$$

$$= \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}}$$

Note:

1. The length + dirⁿ of grad f are independent of the particular choice of Cartesian coordinates
2. ∇f has the direction of maximum increase of f at P.

$$\frac{\partial f}{\partial s} = \nabla f \cdot \vec{b} = |\nabla f|\,|\vec{b}| \cos \gamma$$

where γ is the angle between ∇f and $\vec{b}$. We see that $\frac{\partial f}{\partial s}$ is maximum when $\cos \gamma = 1$ or $\gamma = 0$ and then

$\frac{\partial f}{\partial s} = |\text{grad } f|$. This shows that the direction of maximum increase is in the dir^n of ∇f.

3. Consider a differentiable scalar fct $f(x,y,z)$ in space. $f(x,y,z) = C =$ constant respresents a surface S in space. With C assuming different values, we obtain a family of surfaces which are called the "level surfaces" of f. Since by definition, f has a unique value at each pt in space, it follows that there passes one and only one level surface of f thru each pt.

Let a curve C in space represented by $r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$.

if we now require that C lies on S, then

$$f(x(t), y(t), z(t)) = C$$

$$\Rightarrow \frac{\partial f}{\partial t} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = 0$$

$$\Rightarrow \nabla f \cdot \dot{\vec{r}} = 0$$

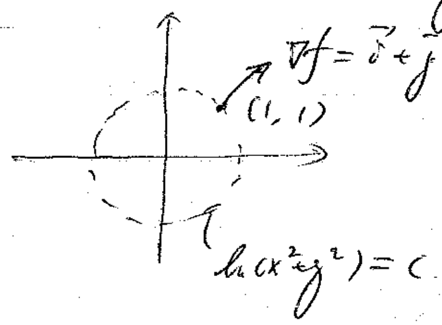
where $\dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$ is tangent to C.

$$\nabla f \cdot \dot{\vec{r}} = 0 \implies \nabla f \perp \dot{\vec{r}}$$
$$\implies \nabla f \text{ is normal to the surface.}$$

<u>Example</u>:

$$f(x,y) = \ln(x^2+y^2)$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \frac{2x}{x^2+y^2}\vec{i} + \frac{2y}{x^2+y^2}\vec{j}$$



@ $x = y = 1$

$$\nabla f = \vec{i} + \vec{j}$$

§ <u>Divergence of a Vector Field</u>

Let $\vec{V}(x,y,z) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a differentiable vector. Then

$$\operatorname{div} \vec{V} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\vec{V}$.

Another common notation is:

$$\operatorname{div} \vec{V} = \nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Note : $\vec{V} \cdot \nabla \neq \nabla \cdot \vec{V}$

Example : Continuity equation in fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2) + \frac{\partial}{\partial z}(\rho v_3) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

ρ = density , $\vec{V}$ = velocity

For incompressible fluid , ρ = constant, it becomes

$$\cancel{\frac{\partial \rho}{\partial t}}^{0} + \rho \frac{\partial v_1}{\partial x} + v_1 \cancel{\frac{\partial \rho}{\partial x}}^{0} + \rho \frac{\partial v_2}{\partial y} + v_2 \cancel{\frac{\partial \rho}{\partial y}}^{0}$$

$$+ \rho \frac{\partial v_3}{\partial z} + v_3 \cancel{\frac{\partial \rho}{\partial z}}^{0} = 0$$

$$\Rightarrow \rho \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) = 0$$

or $\boxed{\nabla \cdot \vec{V} = 0}$ since $\rho \neq 0$

This is the condition of incompressibility

§ Curl of a Vector Field

$$\text{curl } \vec{V} \equiv \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k}$$

is called the curl of the vector fct $\vec{V}$.

Example: Vorticity in fluid mechanics

$$\nabla \times \vec{V} = \vec{\zeta} = 2 \times \vec{\omega}$$

where $\vec{\zeta}$ = vorticity of a fluid element

$$= \zeta_x \vec{i} + \zeta_y \vec{j} + \zeta_z \vec{k}$$

The vorticity components ζ_x, ζ_y & ζ_z are related to the angular speed (twice) of the fluid element about its own axis in the corresponding directions.

If $\nabla \times \vec{V} = 0$, the flow is irrotational

note: $\nabla \times (\nabla f) = \vec{0}$ for any twice differentiable scalar fct f.

hence if a vector fct is the gradient of a scalar fct, its curl is the zero vector. For irrotational flow, $\nabla \times \vec{V} = \vec{\zeta} = 0$, $\vec{V} = \nabla \phi$, i.e

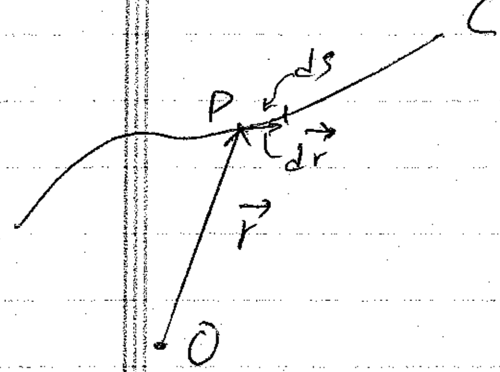
the velocity can be represented as the gradient of a scalar fct ϕ. ϕ is called the velocity potential.

§ <u>Line integrals</u>

Let $\vec{F}$: a vector fct in a 3-D space

C : a space curve

$\vec{r}$: position vector from an origin O to the space curve C



With each pt P of C, we have

$$d\vec{r} = \frac{d\vec{r}}{ds} \cdot ds = \vec{u}\, ds$$

where $\vec{u}$ is a unit tangent vector to C at P. (note: $d\vec{r}$ has as its length ds along C at P and as its dir$^{\underline{n}}$ the direction of the curve at P.)

$$\vec{F} \cdot d\vec{r} = (\vec{F} \cdot \vec{u})\, ds$$

= product of the component of $\vec{F}$ in the direction of C at P and the differential length ds.

Finally, $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{u}\,ds$

is known as the line integral of $\vec{F}$ along C. In particular, if $\vec{F}$ = force acting on a particle, then

$\int_C \vec{F}\cdot d\vec{r}$ = work done by the force in moving the particle along C.

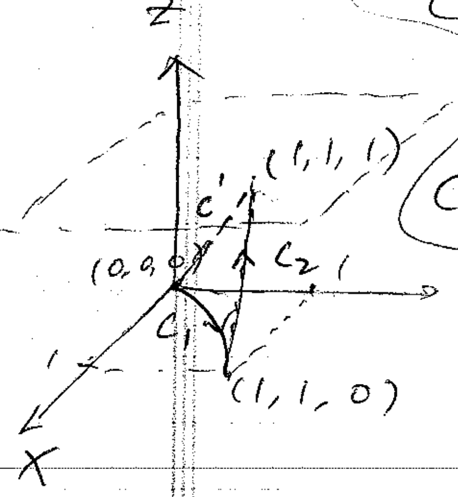
Let $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

& $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + \int Q\,dy + R\,dz$$

Example: $\vec{F} = yz\,\vec{i} + xy\,\vec{j} + xz\,\vec{k}$



C_1 : $(0, 0, 0) \longrightarrow (1, 1, 0)$ along $x = y^2$, $z = 0$

C_2 : $(1, 1, 0) \longrightarrow (1, 1, 1)$ along $x = y = 1$

$$\vec{F} \cdot d\vec{r} = yz\,dx + xy\,dy + xz\,dz$$

along C_1 : $x = y^2$, $z = 0$

$$\therefore \quad dx = 2y\,dy, \quad dz = 0$$

$$\vec{F} \cdot d\vec{r} = y^3\,dy$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 y^3\,dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

along C_2 : $x = y = 1 \Rightarrow dx = dy = 0$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 z\,dz = \frac{1}{2}$$

$$\therefore \int_{C_1 + C_2} \vec{F} \cdot d\vec{r} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Consider an alternate path C' which goes straight from $(0,0,0)$ to $(1,1,1)$ (shown in red on the sketch)

along C' : $x = y = z \Rightarrow dx = dy = dz$

$$\int_{C'} \vec{F} \cdot d\vec{r} = \int_0^1 z^2 dz + z^2 dz + z^2 dz$$

(all in terms of z)

$$= 3\int_0^1 z^2 dz = z^3\Big|_0^1 = 1$$

Note: $\int \vec{F} \cdot d\vec{r}$ depends on the path chosen going from $(0,0,0)$ to $(1,1,1)$.

In some cases, $\int \vec{F} \cdot d\vec{r}$ is path independent. If $\vec{F}$ can be represented as the gradient of a scalar fct, say ϕ, then $\int \vec{F} \cdot d\vec{r}$ is path independent.

<u>Proof</u> : $\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

$$\Rightarrow \int \vec{F} \cdot d\vec{r} = \int \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right)(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \int \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= \int d\phi = \phi_2 - \phi_1$$

where ϕ_1 + ϕ_2 are the values of ϕ at the end pts of the curve. Hence it is independent of path.

So, under what condition when $\vec{F}$ can be represented as $\nabla \phi$? i.e. $\vec{F} = \nabla \phi$.

Since $\nabla \times \nabla \phi \equiv 0$, (vector identity)

ϕ exists iff $\nabla \times \vec{F} = 0$

if $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

$$\nabla \times \vec{F} = i\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + j\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\nabla \times \vec{F} = 0 \implies \boxed{\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \& \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}} \quad (*)$$

i.e. when (*) is satisfied, then a scalar fct $\phi(x,y,z)$ exists s.t.

$$\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = d\phi$$

For a closed curve:

$$\oint_C \vec{F}\cdot d\vec{r} = 0 \quad \text{iff } (*) \text{ is satisfied.}$$

(otherwise it may or may not vanish)

<u>Example</u>: $\vec{F} = \underset{P}{\underset{\shortparallel}{y^2}}\,\vec{i} + \underset{Q}{\underset{\shortparallel}{2(xy+z)}}\,\vec{j} + \underset{R}{\underset{\shortparallel}{2y}}\,\vec{k}$

(i) $\dfrac{\partial P}{\partial y} \overset{?}{=} \dfrac{\partial Q}{\partial x}$ $\underset{2y}{\shortparallel}$ $\underset{2y}{\shortparallel}$ Yes

(ii) $\dfrac{\partial P}{\partial z} \overset{?}{=} \dfrac{\partial R}{\partial x}$ $\underset{0}{\shortparallel}$ $\underset{0}{\shortparallel}$ yes

(iii) $\dfrac{\partial Q}{\partial z} \overset{?}{=} \dfrac{\partial R}{\partial y}$ $\underset{2}{\shortparallel}$ $\underset{2}{\shortparallel}$ yes

$\therefore\ \nabla\times\vec{F} = 0 \ \&\ \vec{F} = \nabla\phi.$

To determine $\phi(x,y,z)$, we have

$$\vec{F}\cdot d\vec{r} = d\phi = y^2 dx + 2(xy+z)dy + 2y\,dz$$

Compared with $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

we have:

$$\frac{\partial \phi}{\partial x} = y^2 \qquad (a)$$

$$\frac{\partial \phi}{\partial y} = 2(xy+z) \qquad (b)$$

$$\frac{\partial \phi}{\partial z} = 2y \qquad (c)$$

(a) gives: $\phi = xy^2 + f(y,z)$

$$\Rightarrow \frac{\partial \phi}{\partial y} = 2xy + \frac{\partial f(y,z)}{\partial y}$$

Comparing with (b) gives:

$$\frac{\partial f(y,z)}{\partial y} = 2z$$

$$\Rightarrow f(y,z) = 2yz + g(z)$$

$$\therefore \phi = xy^2 + 2yz + g(z)$$

$$\Rightarrow \frac{\partial \phi}{\partial z} = 2y + g'(z)$$

Comparing with (c) gives,

$$g'(z) = 0 \Rightarrow g(z) = c$$

$$\therefore \boxed{\phi = xy^2 + 2yz + c}$$