MAE 3360 ENGINEERING ANALYSIS II

SUMMER 2009

DEPARTMENT OF MECHANICAL
AND
AEROSPACE ENGINEERING

UNIVERSITY OF TEXAS AT ARLINGTON

# Exam #2

July 29, 2009
9:00-10:45 am

CLOSED BOOKS
Non-programmable calculators are allowed

This exam has a total of 7 pages.

## Show all your work
## Illegible writings or incomplete explanations will result in loss of points

LAST NAME : Sol'n

FIRST NAME : ____________________

1.(20 pts) Find the general solution of: $\frac{d^2y}{dx^2} + y = \sec x$

$y_h$ : $e^{mx} \Rightarrow m^2+1=0 \Rightarrow m = \pm i$

$\Rightarrow y_h = c_1 \cos x + c_2 \sin x$

$y_1$ $y_2$

$y_p$ : (variation of parameters)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$y_p = u y_1 + v y_2$$

$$u' = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = \frac{-\sin x}{\cos x}$$

$$\Rightarrow u = \int -\frac{\sin x}{\cos x}\,dx = \int \frac{d\cos x}{\cos x} = \ln(\cos x)$$

$$v' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = 1 \Rightarrow v = x$$

$$\therefore y_p = u y_1 + v y_2 = \cos x \ln(\cos x) + x \sin x$$

$$\boxed{y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln(\cos x) + x \sin x}$$

2.(20 pts) Find the general solution of: $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=\frac{1}{x}$

$$y''-\frac{4}{x}y'+\frac{6}{x^2}y=\frac{1}{x^3}$$

$y_h$ : Let $y_h=x^m \Rightarrow m(m-1)-4m+6=0$

$\Rightarrow m^2-5m+6=0$

$\Rightarrow m=2,\ 3$

$\therefore y_h=c_1\underset{y_1}{x^2}+c_2\underset{y_2}{x^3}$

$y_p$ : $y_p=uy_1+vy_2$ (variation of parameters)

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}=x^4$$

$$u'=\frac{1}{x^4}\begin{vmatrix} 0 & x^3 \\ 1/x^3 & 3x^2 \end{vmatrix}=-\frac{1}{x^4}\Rightarrow u=\frac{1}{3}x^{-3}$$

$$v'=\frac{1}{x^4}\begin{vmatrix} x^2 & 0 \\ 2x & 1/x^3 \end{vmatrix}=\frac{1}{x^5}\Rightarrow v=-\frac{1}{4}x^{-4}$$

$$\therefore y_p=ux^2+vx^3=\frac{1}{3}x^{-3}x^2-\frac{1}{4}x^{-4}x^3=\frac{1}{12x}$$

$$\therefore \boxed{y=y_h+y_p=c_1x^2+c_2x^3+\frac{1}{12x}}$$

3.(20pts) Consider the damped spring-mass system whose motion is governed by

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=10\sin t. \;; \quad y(0)=1\,,\; y'(0)=0$$

(a) Determine whether the motion is underdamped, overdamped or critically damped.

(b) Determine the amplitude of the steady-state oscillation.

(a) $m^2+3m+2=0 \Longrightarrow m=-1,\,-2$

or $y_h = C_1e^{-t}+C_2e^{-2t}$

Clearly the system is overdamped, i.e. no sinusoidal oscillations!

(b) $y_p = A\sin t + B\cos t$

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$

Sub. into D.E. gives:

$(-A\sin t - B\cos t)$
$+3(A\cos t - B\sin t)$
$+2(A\sin t + B\cos t)$
$= 10\sin t$

Equating coeffs:

$-A-3B+2A=10$

& $-B+3A+2B=0$

$\Longrightarrow A=1$ & $B=-3$

$\therefore y_p = \sin t - 3\cos t$

$\Longrightarrow$ Amplitude of steady-state oscillation $=\sqrt{1^2+3^2}=\sqrt{10}$

Note: I.C.'s are not needed as the $y_h$ is not needed for the steady state sol$^{\underline{n}}$

4.(20pts) A mass weighing 32 pounds is attached to a spring whose spring constant is 5 lb/ft. Initially the mass is released from rest from a point 1 feet below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to twice the instantaneous velocity. Determine the equation of motion if the mass is driven by an external force, in pounds, equal to 10 cos 3t. **Explain clearly the formulation of the problem.**

$W = 32\ lb \Rightarrow m = 1\ slug$

$k = 5\ lb/ft$

$\beta = 2$

$\Rightarrow m\ddot{x} + \beta\dot{x} + kx = 10\cos 3t$ (in pounds)

$m$: 1 slug, $\ddot{x}$: ft/s², $\beta$: 2, $\dot{x}$: ft/s, $k$: 5 lb/ft, $x$: ft

$\Rightarrow \boxed{\ddot{x} + 2\dot{x} + 5x = 10\cos 3t}$ (1)

I.C. $x(0) = 1$ & $\dot{x}(0) = 0$

$x_h$: $\ddot{x}_h + 2\dot{x}_h + 5x_h = 0 \Rightarrow m^2 + 2m + 5 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{2^2 - 4\cdot 5}}{2}$

$= -1 \pm 2i$

$\therefore \boxed{x_h = e^{-t}(C_1\cos 2t + C_2\sin 2t)}$

$x_p$: $x_p = A\cos 3t + B\sin 3t$

$\dot{x}_p = -3A\sin 3t + 3B\cos 3t$

$\ddot{x}_p = -9A\cos 3t - 9B\sin 3t$

Sub. into Eq (1) gives

$(-9A + 6B + 5A)\cos 3t + (-9B - 6A + 5B)\sin 3t = 10\cos 3t$

Equating coeffs: $6B - 4A = 10$, $-4B - 6A = 0$ $\Rightarrow A = -\frac{10}{13}$, $B = \frac{15}{13}$

$\therefore \boxed{x_p = -\frac{10}{13}\cos 3t + \frac{15}{13}\sin 3t}$

$\boxed{x = x_h + x_p = e^{-t}(C_1\cos 2t + C_2\sin 2t) + \frac{15}{13}\sin 3t - \frac{10}{13}\cos 3t}$

Apply I.C. (i) $x(0) = 1 \Rightarrow 1 = C_1 - \frac{10}{13} \Rightarrow \boxed{C_1 = 1\frac{10}{13}}$

(ii) $\dot{x}(0) = 0 \Rightarrow \boxed{C_2 = -\frac{11}{13}}$

5a.(10pts) Using the fourth-order Runge-Kutta formula, with $\Delta t = 0.1$, determine the solution of y, up to the fourth decimal place, at $t = 0.1$ for the following initial value problem:

$$\frac{dy}{dt} = 1 + y^2 \; ; \quad y(0) = 0$$

**Show clearly details of all computations.**

$$f(t, y) = 1 + y^2$$

$$t_0 = 0, \quad y_0 = 0$$

$$h = \Delta t = 0.1$$

$$k_1 = f(t_0, y_0) = 1 + y_0^2 = 1$$

$$k_2 = f\left(t_0 + \frac{\Delta t}{2}, \; y_0 + \frac{k_1 h}{2}\right) = f(0.05, 0.05) = \cancel{1.00}$$

$$= 1 + 0.05^2 = 1.0025$$

$$k_3 = f\Big(0.05, \underbrace{y_0}_{0} + \underbrace{\frac{k_2 h}{2}}_{\frac{(1.0025)(0.1)}{2}}\Big) = 0.050125$$

$$k_4 = f(0.1, \; y_0 + \underbrace{k_3 h}_{(1.002513)(0.1)}) = 1.0100$$

$$y_1 = y_0 + \frac{0.1}{6}(\underbrace{k_1}_{1} + \underbrace{2k_2}_{2(1.0025)} + \underbrace{2k_3}_{2(1.002513)} + \underbrace{k_4}_{1.0100})$$

$$= \boxed{0.1003}$$

5b.(10pts)Using the Euler's method, with $\Delta t = 0.1$, determine the solution of y, up to the fourth decimal place, at $t = 0.2$ for the following initial value problem:

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = 0 \quad ; \quad y(0) = 1, \ \frac{dy}{dt}(0) = 3$$

**Show details of all computations and enter the values of y in the table below.**

| t | 0.1 | 0.2 |
|---|---|---|
| y | 1.3 | 1.58 |

$y'' + t y' + 2y = 0$

$y' = u \Rightarrow u' = -tu - 2y$

$t_0 = 0, \ y_0 = 1 \ ; \ y_0' = 3 \Rightarrow u_0 = 3$

$\Delta t = 0.1$

$y_1 = y_0 + \Delta t\, u_0 = 1 + (0.1)(3) = 1.3$

$u_1 = u_0 + \Delta t\,(-t_0 u_0 - 2y_0)$

$= 3 + 0.1\,(-(0)(3) - 2(1)) = 2.8$

$y_2 = y_1 + \Delta t\, u_1 = 1.3 + (0.1)(2.8)$

$= 1.58$