SI Exam Review - II

MAE 3360

Dt: 02/27/2008.

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* Solution of linear O.D.E. with constant co-efficients:-

General Approach:-

→ First obtain the ~~correct~~ the ~~into~~ characteristic equation from the given ODE with constant co.eff

→ Solve the quadratic characteristic equation

3 possibilities:- ...

(I) two distinct real roots

Sol^n: $y = A_1 e^{m_1 t} + A_2 e^{m_2 t}$ — m_1, m_2 - roots, A_1, A_2 - const.

(II) Repeated roots

Sol^n: $y = A_1 e^{mt} + t A_2 e^{mt}$ — m - repeated roots

(III) two complex conjugate roots — $m = a + ib$

Sol^n: $y = e^{ax}(A_1 \cos bx + A_2 \sin bx)$

Example: $\ddot{y} + 10\dot{y} + 21y = 0$.

Sol^n: The given ODE is $\ddot{y} + 10\dot{y} + 21y = 0$

∴ The characteristic equation from the given ODE is,

$$m^2 + 10m + 21 = 0$$

$$\therefore (m+7)(m+3) = 0$$

$$\therefore m = -7, -3$$ two distinct real roots.

$\therefore$ sol^n is

$$\boxed{y = A_1 e^{-7t} + A_2 e^{-3t}}$$ ← solution.

Ex:- (2) $y'' + 4y' + 5y = 0$

→ The characteristic equation from the given ODE is,

$$m^2 + 4m + 5 = 0$$

$$\therefore m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$\boxed{m = -2 \pm i}$$

→ complex conjugate roots

$\therefore$ sol^n: $\boxed{y = \cancel{A} e^{-2x} (c_1 \cos x + c_2 \sin x)}$

Ex:- (3) $\frac{d^2P}{dt^2} - 18 \frac{dP}{dt} + 81P = 0$

→ From the given 2nd order linear ODE with constant co-efficient, the characteristic equation will be,

$$m^2 - 18m + 81 = 0$$

$$\therefore (m-9)^2 = 0$$

$\therefore$ $m = 9$ a double root.

So, now the solution is,

$$\boxed{P = c_1 e^{9t} + t c_2 e^{9t}}$$ ← solution.

★ Equidimensional Equations:- (Linear ODE with Variable Co.eff.)

If the given ODE is of the general form like,

$$x^n \frac{d^n y}{dx^n} + b x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_{n-1} x \frac{dy}{dx} + b_n y = 0$$

all b's are constant.

→ To solve such equation, take $y = x^m$.

and then find out the $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \cdots \frac{d^n y}{dx^n}$ and

Plug-in those values to the given ODE.

→ Find out the characteristic equation for the above form and solve it using quadratic equation Solution.

→ Substitute values of roots to the trial solution
~~Exam~~ and get the final Solution.

→ For Examples of such ODE refer to the ~~exam~~ SI notes of 13th February, 2008.

* Solution of Non-Homogeneous Equations:-

For Non-Homogeneous ODE equation, solution, ...
$L_n(y) = \phi(x)$

1) Find out the solution of Homogeneous Equation (y_h) by taking $L_n(y)=0$.

2) Find out the particular solution (y_p) by using method of Undetermined co-efficients as for the cases where family can be found easily for non-Homogeneous terms.

3) General solution will be the sum of the Homogeneous & Particular solution.

$$y = y_h + y_p$$

* Method of Undetermined co-efficients for (y_p):-

→ First decide the Family [look table in textbook for some Families]

→ Take trial y_p from the choosen Family

→ Plug-in it in the given ODE and by using comparison find out the co-efficients (unknown) of y_p

Note: If you find the exact same term in the y_p as it is in y_h then modify the y_p until it doesn't contain the exact same term of y_h.

Example: $y' - 5y = x^2 e^x - x e^{5x}$

Sol^n: ① Find out y_h:

$\therefore y' - 5y = 0$

$\Rightarrow$ characteristic equation,

$m - 5 = 0 \Rightarrow \boxed{m = 5}$

$$\boxed{\therefore y_h = A_1 e^{5x}}$$

② Find y_p:

From non-homogeneous terms,

Family for $x^2 e^x$

$y_{p_1} = e^x (Ax^2 + Bx + C)$

For $x e^{5x}$ part,

$y_{p_2} = e^{5x} (Dx + E)$

Now, the y_{p_2} contains the term of y_h $\{ \because E e^{5x} \}$

So we need to modify it,

$\therefore y_{p_2} = e^{5x} (Dx^2 + Ex)$

Now its free from the terms of y_h.

So, $y_p = y_{p_1} + y_{p_2}$

$\therefore y_p = e^x (Ax^2 + Bx + C) + e^{5x} (Dx^2 + Ex)$ — (*)

Now, $y_p' = e^x (2Ax + B) + e^x (Ax^2 + Bx + C) + e^{5x} (2Dx + E)$

$+ 5e^{5x} (Dx^2 + Ex)$

Now from given ODE

$$y_p' - 5y_p = x^2 e^x - x e^{5x}$$

$$\therefore e^x(2Ax+B) + e^x(Ax^2+Bx+c) + e^{5x}(2Dx+E)$$
$$+ 5e^{5x}(Dx^2+Ex) - 5e^x(Ax^2+Bx+c) - 5e^{5x}(Dx^2+Ex)$$
$$= x^2 e^x - x e^{5x}$$

$$\therefore 2Axe^x + Be^x - 4e^x(Ax^2+Bx+c) + 2Dxe^{5x} + Ee^{5x}$$
$$= x^2 e^x - x e^{5x}$$

$$\therefore 2Axe^x + Be^x - 4Ax^2e^x - 4Bxe^x - 4Ce^x + 2Dxe^{5x}$$
$$+ Ee^{5x} = x^2 e^x - x e^{5x}$$

$$\therefore (2A-4B)xe^x - 4Ax^2e^x + (B-4C)e^x + 2Dxe^{5x} + Ee^{5x}$$
$$= x^2 e^x - x e^{5x}$$

Now, comparing both sides we have,

$2A - 4B = 0$ $\quad$ $B - 4C = 0$ $\quad$ $\boxed{E = 0}$

$-4A = 1$ $\quad$ $2D = -1$

$\Rightarrow \boxed{A = -1/4}$ $\quad$ $\boxed{D = -1/2}$

Now, $2(-1/4) - 4B = 0$

$\therefore -4B = 1/2$

$\Rightarrow \boxed{B = -1/8}$

Now, $-1/8 - 4C = 0$

$\therefore -4C = 1/8$

$\Rightarrow \boxed{C = -1/32}$

So, Now,

$$y_p = e^{x}\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) + e^{5x}\left(-\frac{1}{2}x^2\right) \quad \{\because (*)\}$$

$$\boxed{\therefore y_p = e^{x}\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2 e^{5x}}$$

Now, general solution,

$$y = y_h + y_p$$

$$\boxed{\therefore y = A_1 e^{5x} + e^{x}\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2 e^{5x}}$$

Solution.