MAE 3360 ENGINEERING ANALYSIS

Fall 2006

DEPARTMENT OF MECHANICAL
AND
AEROSPACE ENGINEERING

UNIVERSITY OF TEXAS AT ARLINGTON

Exam #2

CLOSED BOOKS and NO CALCULATORS
(Laplace transform table is provided)

Nov 10, 2006
Time Limit : 50 min

(This exam has 6 pages)

Show all your work
Illegible writings or incomplete explanations will result in loss of points

LAST NAME : Sol^{N}

FIRST NAME : ____________

1.(30 pts) Find the general solution by the variation of parameters method to:

$$\frac{d^2y}{dx^2} - 9y = \frac{9x}{e^{3x}} \qquad (\text{ref: H.W. \#4 Ex 3.5 \#10})$$

$$y_h: \quad y'' - 9y = 0 \implies m^2 - 9 = 0 \implies m = \pm 3$$

$$\therefore \; y_1 = e^{3x} \; ; \; y_2 = e^{-3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6$$

$$y = u y_1 + v y_2 \; ; \quad f(x) = 9x e^{-3x}$$

$$\left. \begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1' + v' y_2' &= f(x) \end{aligned} \right\} \implies u' = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} \Big/ W$$

$$= \begin{vmatrix} 0 & e^{-3x} \\ 9xe^{-3x} & -3e^{-3x} \end{vmatrix} \Big/ -6$$

$$u' = \frac{3}{2} x e^{-6x} \implies u = \int \frac{3}{2} x e^{-6x}\, dx$$

Integrating by parts: $u = \frac{3}{2} \int \left(x \frac{d e^{-6x}}{-6} \right) = -\frac{1}{4}\left(x e^{-6x} - \int e^{-6x} dx \right)$

$$= -\frac{1}{4}\left(x e^{-6x} + \frac{1}{6} e^{-6x} \right)$$

$$v' = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} \Big/ W = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 9xe^{-3x} \end{vmatrix} \Big/ -6 = -\frac{3}{2} x$$

$$\therefore \; v = \int -\frac{3}{2} x \, dx = -\frac{3}{4} x^2$$

$$\therefore \; y_p = u y_1 + v y_2 = -\frac{1}{4}\left(x e^{-6x} + \frac{1}{6} e^{-6x} \right) e^{3x} - \frac{3}{4} x^2 e^{-3x} = -\frac{1}{24} e^{-3x} - \frac{1}{4} x e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

$$y = y_h + y_p = C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{24} e^{-3x} - \frac{1}{4} x e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

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2a.(10 pt) Find the inverse Laplace transform of $\frac{s+1}{s^2-2s+5}$ (ref: H.W. #6 Ex 4.2 #42)

$$\frac{s+1}{s^2-2s+5} = \frac{s+1}{(s-1)^2+2^2}$$

$$= \frac{s-1}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2}$$

$$\mathcal{L}^{-1} = \boxed{e^{t}\cos 2t + e^{t}\sin 2t}$$

2b.(10 pts) Find the inverse Laplace transform of $\frac{s^2+1}{s(s-1)(s+1)(s-2)}$ (ref: H.W. #6 Ex 4.2 #26)

$$= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}$$

$$A = \left.\frac{s^2+1}{(s-1)(s+1)(s-2)}\right|_{s=0} = \frac{1}{(-1)(1)(-2)} = \frac{1}{2}$$

$$B = \left.\frac{s^2+1}{s(s+1)(s-2)}\right|_{s=1} = \frac{2}{1(2)(-1)} = -1$$

$$C = \left.\frac{s^2+1}{s(s-1)(s-2)}\right|_{s=-1} = \frac{2}{(-1)(-2)(-3)} = -\frac{1}{3}$$

$$D = \left.\frac{s^2+1}{s(s-1)(s+1)}\right|_{s=2} = \frac{5}{(2)(1)(3)} = \frac{5}{6}$$

$$\therefore \boxed{\mathcal{L}^{-1} = \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}}$$

3.(30 pts) Find the solution y(t) using the Laplace transform method to:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-4t} \quad \text{with } y(0) = 1 \text{ and } \frac{dy}{dt}(0) = 5.$$

(ref: Example 5 in p. 204 of text)

$$s^2 Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=5} - 3\big(sY(s) - \underbrace{y(0)}_{=1}\big) + 2Y(s) = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2)Y - s - 5 + 3 = \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \left((s+2) + \frac{1}{s+4}\right) \Big/ \underbrace{(s^2 - 3s + 2)}_{=(s-1)(s-2)}$$

$$= \frac{(s+2)(s+4) + 1}{(s+4)(s-1)(s-2)}$$

$$= \frac{s^2 + 6s + 9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$A = \left.\frac{s^2 + 6s + 9}{(s-1)(s-2)}\right|_{s=-4} = \frac{16 - 24 + 9}{(-5)(-6)} = \frac{1}{30}$$

$$B = \left.\frac{s^2 + 6s + 9}{(s+4)(s-2)}\right|_{s=1} = \frac{1 + 6 + 9}{(5)(-1)} = -\frac{16}{5}$$

$$C = \left.\frac{s^2 + 6s + 9}{(s+4)(s-1)}\right|_{s=2} = \frac{4 + 12 + 9}{(6)(1)} = \frac{25}{6}$$

$$\therefore \boxed{y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}}$$

$g = 32 \text{ ft/s}^2$

4. (20 pts) A mass weighing 64 pounds stretches a spring of 0.32 foot. The mass is initially relaeased from a point 8 inches above the equilibrium position with a downward velocity of 5 ft/s. What are the amplitude and period of motion? (ref. Prob. 11 on p. 162)

$$my'' + ky = 0 \quad ; \quad k = \frac{64 \text{ lb}}{0.32 \text{ ft}} = 200 \frac{\text{lb}}{\text{ft}}$$

$$m = \frac{64 \text{ lb}}{g} = \frac{64}{32} = 2 \text{ slug}$$

$$\therefore \quad 2y'' + 200y = 0$$

$$\text{or} \quad y'' + 100y = 0$$

$$\& \quad \boxed{y = A\cos 10t + B\sin 10t}$$

B.C.

(i) $y(0) = -\frac{2}{3}$ (ii) $y'(0) = 5$

(i) $y(0) = -\frac{2}{3} = A$

(ii) $y'(0) = 5 = 10B \Rightarrow B = \frac{1}{2}$

$$\therefore \quad y = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t$$

$$\text{Amplitude} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \boxed{\frac{5}{6}}$$

$$\text{period} = \frac{2\pi}{10} = \boxed{\frac{\pi}{5}} \text{ seconds}$$

Laplace Transforms

	$f(x)$	$F(s) = \mathscr{L}\{f(x)\}$
1.	1	$\frac{1}{s} \quad (s>0)$
2.	x	$\frac{1}{s^2} \quad (s>0)$
3.	$x^{n-1} \quad (n = 1, 2, \ldots)$	$\frac{(n-1)!}{s^n} \quad (s>0)$
4.	$\sqrt{x}$	$\frac{1}{2}\sqrt{\pi}\, s^{-3/2} \quad (s>0)$
5.	$1/\sqrt{x}$	$\sqrt{\pi}\, s^{-1/2} \quad (s>0)$
6.	$x^{n-1/2} \quad (n = 1, 2, \ldots)$	$\frac{(1)(3)(5)\cdots(2n-1)\sqrt{\pi}}{2^n} s^{-n-1/2} \quad (s>0)$
7.	e^{ax}	$\frac{1}{s-a} \quad (s>a)$
8.	$\sin ax$	$\frac{a}{s^2+a^2} \quad (s>0)$
9.	$\cos ax$	$\frac{s}{s^2+a^2} \quad (s>0)$
10.	$\sinh ax$	$\frac{a}{s^2-a^2} \quad (s>\lvert a\rvert)$
11.	$\cosh ax$	$\frac{s}{s^2-a^2} \quad (s>\lvert a\rvert)$
12.	$x \sin ax$	$\frac{2as}{(s^2+a^2)^2} \quad (s>0)$
13.	$x \cos ax$	$\frac{s^2-a^2}{(s^2+a^2)^2} \quad (s>0)$
14.	$x^{n-1}e^{ax} \ (n = 1, 2, \ldots)$	$\frac{(n-1)!}{(s-a)^n} \quad (s>a)$
15.	$e^{bx} \sin ax$	$\frac{a}{(s-b)^2+a^2} \quad (s>b)$
16.	$e^{bx} \cos ax$	$\frac{s-b}{(s-b)^2+a^2} \quad (s>b)$