Solutions to HW#6 SP07

#1.

An idealized velocity field is given by the formula

$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$

Is this flow field steady or unsteady? Is it two- or three- dimensional? At the point (x, y, z) = (-1, 1, 0), compute *(a)* The acceleration vector and *(b)* any unit vector normal to the acceleration.

$V = 4tx\,\hat{i} - 2t^2y\,\hat{j} + 4xz\,\hat{k}$ (It is unsteady, 3-D flow)

$a = \dfrac{dV}{dt} = \dfrac{\partial V}{\partial t} + (V\cdot\nabla)V$ —— I

$\dfrac{\partial V}{\partial t} = 4x\,\hat{i} - 4yt\,\hat{j}$ —— II

$$(V\cdot\nabla)V = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)V$$

$$= 4tx\frac{\partial V}{\partial x} - 2t^2y\frac{\partial V}{\partial y} + 4xz\frac{\partial V}{\partial z}$$

$$= (4tx)(4t\,\hat{i} + 4z\,\hat{k}) - (2t^2y)(-2t^2\,\hat{j}) + (4xz)(4x\,\hat{k})$$

$$= 16xt^2\,\hat{i} + 16xzt\,\hat{k} + 4t^4y\,\hat{j} + 16x^2z\,\hat{k} \quad \text{—— III}$$

Substitute II, III in I.

$$a = 4x\,i - 4yt\,j + 16xt^2\,i + 16xzt\,k + 4t^4y\,j + 16x^2z\,k.$$

$$= 4x(1+4t^2)i + 4yt(t^3-1)j + 16xz(x+t)k.$$

@ point $(x, y, z) = (-1, 1, 0)$

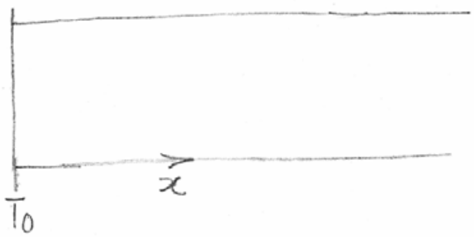
(a) $acc = -4(1+4t^2)\,\hat{i} + 4t(t^3-1)\,\hat{j}$ Ans.

(b) Since acc is in x-y plane, z-axis is normal to x-y plane and therefore unit vector normal to acc is $\hat{k}$

#2.

The temperature T, in a long tunnel is known to vary approximately as $T = T_0 - \alpha e^{-x/L} \sin(2\pi t/\tau)$, where T_0, α, L and τ are constants, and x is measured from the entrance. A particle moves into the tunnel with a constant speed, U. Obtain an expression for the rate of change of temperature experienced by the particle. What are the dimensions of this expression?

Given : $T = T_0 - \alpha e^{-\frac{x}{L}} \sin\left(\frac{2\pi t}{\tau}\right)$



$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (V \cdot \nabla)T$$

$$\frac{\partial T}{\partial t} = 0 - \alpha e^{-x/L} \cos\frac{2\pi t}{\tau} \cdot \frac{2\pi}{\tau} = -\frac{2\pi}{\tau}\alpha e^{-x/L} \cos\frac{2\pi t}{\tau} \quad \text{— (I)}$$

$$(V \cdot \nabla)T = U\frac{\partial}{\partial x}\left(T_0 - \alpha e^{-x/L} \sin\frac{2\pi t}{\tau}\right)$$

$$= U\left(0 - \alpha \sin\frac{2\pi t}{\tau} \cdot e^{-x/L} \cdot -\frac{1}{L}\right.$$

$$= \frac{\alpha U}{L} e^{-x/L} \sin\frac{2\pi t}{\tau} \quad \text{— (II)}$$

substitute (I) and (II) in $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (V \cdot \nabla)T$

$$\frac{DT}{Dt} = -\frac{2\pi\alpha}{\tau} e^{-x/L} \cos\frac{2\pi t}{\tau} + \frac{\alpha U}{L} e^{-x/L} \sin\frac{2\pi t}{\tau}$$

$$\frac{DT}{Dt} = \alpha e^{-x/L}\left[\frac{U}{L}\sin\frac{2\pi t}{\tau} - \frac{2\pi}{\tau}\cos\frac{2\pi t}{\tau}\right] \quad \text{Ans.}$$

Dimensions: $\frac{\theta}{T} = \alpha \cdot \text{Dimensionless} \cdot \left[\frac{1}{T} \cdot \text{Dimensionless} - \frac{1}{T} \cdot \text{Dimensionless}\right]$

so $\frac{\theta}{T} = \frac{\alpha}{T} - \frac{\alpha}{T}$

so $\frac{\theta}{T} = \left(\frac{\theta}{T}\right) - \left(\frac{\theta}{T}\right)$

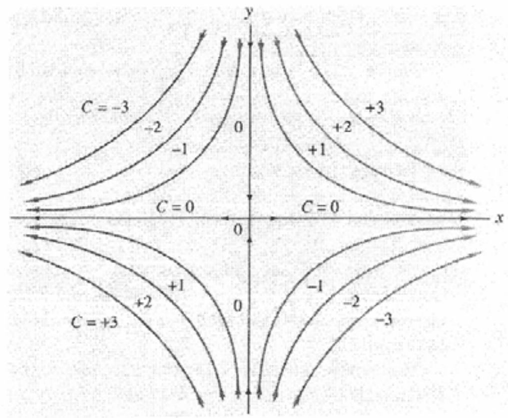
so for dimensional homogeneity α must have dimensions of temp. i.e. θ

#3.

The velocity field near a stagnation point (see Fig.) may be written in form $u = \frac{U_0 x}{L}$, $v = -\frac{U_0 y}{L}$ U_o and L are constants.

(a) Show that the acceleration vector is purely radial.

(b) For the particular case L = 1.5 m, if the acceleration at (x, y) = (1 m, 1 m) is 25 m/s^2, what is the value of U_0?



Velocity V :

$$V = \frac{U_0 x}{L} i - \frac{U_0 y}{L} j$$

$$a = \cancel{\frac{\partial V}{\partial t}} + (V \cdot \nabla) V$$

0 (steady flow)

So $a = (V . \nabla) V$

$$= \frac{U_0 x}{L} \frac{\partial V}{\partial x} + \frac{U_0 y}{L} \frac{\partial V}{\partial y}$$

$$= \frac{U_0^2}{L^2} (x i + y j)$$

since U_0, L are constt so $\frac{U_0^2}{L^2} = C$

$a = C(x i + y j)$ in polar coordinates $x = R\cos\theta$, $y = R\sin\theta$.

$$a = C(R\cos\theta \, i + R\sin\theta \, j)$$

$$|a| = \sqrt{C^2(R^2\cos^2\theta + R^2\sin^2\theta)} = \sqrt{(CR)^2(\cos^2\theta + \sin^2\theta)}$$

(a) So $|a| = CR$ i.e. Radial Ans.

(b) $(x, y) = (R\cos\theta, R\sin\theta) = (1, 1)$

$$|a| = \sqrt{C^2(R^2\cos^2\theta + R^2\sin^2\theta} = \sqrt{C^2(1+1)} = \sqrt{2}C$$

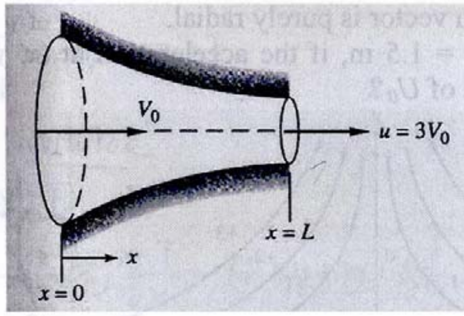
$$25 = \sqrt{2} C \Rightarrow C = \frac{25}{\sqrt{2}}$$

$$\frac{U_0^2}{L^2} = \frac{25}{\sqrt{2}} \Rightarrow U_0 = \sqrt{\frac{25 L^2}{\sqrt{2}}} = \sqrt{\frac{25 \times 2.25}{\sqrt{2}}} = 6.31 \text{ m/s}$$

Ans

#4.

Assume that the flow in the converging nozzle shown has the form $\mathbf{V} = V_0[1+(2x)/L]\mathbf{i}$. Compute *(a)* the fluid acceleration at $x = L$ and *(b)* the time required for a fluid particle to travel from $x = 0$ to $x = L$.



$$V = V_0\left[1+\frac{2x}{L}\right]\hat{i}$$

$$a = \frac{dV}{dt} = \cancel{\frac{\partial V}{\partial t}}^{\,0} + (V\cdot\nabla)V = (V\cdot\nabla)V$$

$$= V_0\left(1+\frac{2x}{L}\right)\frac{\partial}{\partial x}\left[V_0\left(1+\frac{2x}{L}\right)\right]$$

$$= V_0\left(1+\frac{2x}{L}\right)\cdot\frac{2V_0}{L} = \frac{2V_0^2}{L}\left(1+\frac{2x}{L}\right)\hat{i}$$

(a) at $x = L$

$$a = \frac{2V_0^2}{L}\left(1+2\frac{L}{L}\right) = \frac{6V_0^2}{L}\,\hat{i} \quad \underline{\text{Ans}}$$

(b) $$V = \frac{dx}{dt} = V_0\left(1+\frac{2x}{L}\right) = \frac{V_0}{L}(L+2x)$$

$$\frac{dt}{dx} = \frac{L}{V_0}\cdot\frac{1}{L+2x}$$

$$\int_0^t dt = \frac{L}{V_0}\int_0^L \frac{1}{L+2x}\,dx = \frac{L}{V_0}\left[\frac{\ln(L+2x)}{2}\right]_0^L$$

$$t = \frac{L}{2V_0}\left[\ln 3L - \ln L\right] = \frac{L}{2V_0}\left(\ln\frac{3\cancel{L}}{\cancel{L}}\right)$$

so $t = \dfrac{L}{2V_0}\ln 3$ $\underline{\text{Ans}}$

5.

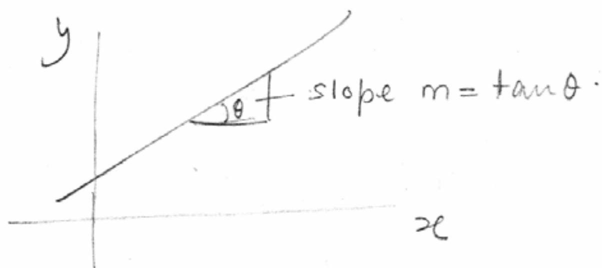
A velocity field is given by $u = V \cos \theta$, $v = V \sin \theta$, and $w = 0$, where V and θ are constants. Derive a formula for the streamlines of this flow.

streamline has $\frac{dy}{dx} = \frac{v}{u} = \frac{V \sin\theta}{V \cos\theta} = \tan\theta$.

therefore $\frac{dy}{dx} = \tan\theta$

$\Rightarrow \quad y = (\tan\theta)\, x + c$ Ans

similar to $y = mx + c$.



#6.

A steady, two-dimensional velocity field is given by $u = 0.5 + 0.8\,x$, $v = 1.5 - 0.8\,y$, where x *and* y coordinates are in meters and the magnitude of velocity is in m/s. Derive a formula for the streamlines of this flow and sketch them in the right half of flow (x > 0).

Eqn. of streamline :

$$\frac{dy}{dx} = \frac{v}{u} = \frac{1.5 - 0.8y}{0.5 + 0.8x}.$$

$$\Rightarrow \int \frac{dy}{1.5 - 0.8y} = \int \frac{dx}{0.5 + 0.8x} + C.$$

$$\frac{\ln(1.5 - 0.8y)}{-0.8} = \frac{\ln(0.5 + 0.8x)}{0.8} + \ln C.$$

$$\ln(1.5 - 0.8y) = -\ln(0.5 + 0.8x) - 0.8 \ln C.$$

$$\ln(1.5 - 0.8y)(0.5 + 0.8x) = -\ln C$$

$$(1.5 - 0.8y)(0.5 + 0.8x) = -C$$

$$1.5 - 0.8y = \frac{-C}{0.5 + 0.8x}$$

$$\Rightarrow y = \frac{C}{0.8(0.5 + 0.8x)} + 1.875$$ Ans