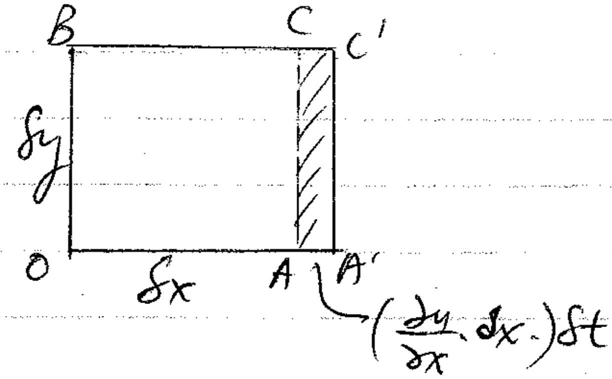
MAE 2314 Fluid Mechanics  Fall 2008  Albert Y. Tong

# Differential Analysis of Fluid Flows

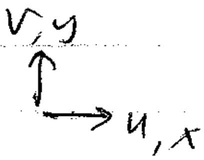
## § Fluid Element Kinematics

General Motion = Translation + Linear Deformation + Rotation + Angular Deformation

Linear Motion & Deformation:




$$\delta V = \delta x \delta y \delta z$$




$$\text{change in } \delta V = \left(\frac{\partial u}{\partial x}\delta x \delta t\right)\delta y \delta z$$

$\Rightarrow$ rate of change in $\delta V$ per unit volume (due to stretching in the x-dir$^n$) $= \frac{1}{\delta V}\frac{d(\delta V)}{dt} = \lim_{\delta t \to 0}\left(\frac{\partial u}{\partial x}\frac{\delta t}{\delta t}\right) = \frac{\partial u}{\partial x}$

Similarly, we have $\frac{\partial v}{\partial y}$ & $\frac{\partial w}{\partial z}$ due to stretching in the y- & z- direction, respectively.

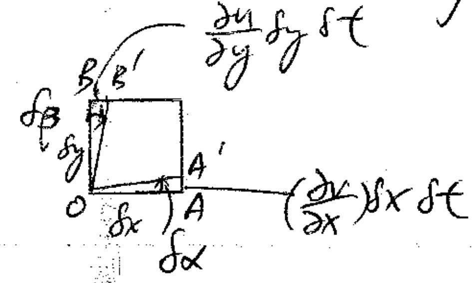
$\therefore$ The total rate of change in $\delta V$ per unit volume $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$= \nabla \cdot \vec{V}$

Volume Dilatation Rate

For incompressible flow, $\nabla \cdot \vec{V} = 0$

**Angular Motion & Deformation:**




$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t}$$

$$\delta\alpha \approx \frac{\partial v}{\partial x}\delta x\,\delta t / \delta x$$

$$\Rightarrow \omega_{OA} = \lim_{\delta t \to 0}\left(\frac{\partial v}{\partial x}\frac{\delta t}{\delta t}\right)$$

$$= \frac{\partial v}{\partial x} \quad \text{(counter clockwise)}$$

Similarly,

$$\omega_{OB} = \lim_{\delta t \to 0}\left(\frac{\delta\beta}{\delta t}\right)$$

& $$\delta\beta \approx \frac{\partial u}{\partial y}\delta y\,\delta t / \delta y = \frac{\partial u}{\partial y}\delta t$$

$$\Rightarrow \omega_{OB} = \lim_{\delta t \to 0}\left(\left(\frac{\partial u}{\partial y}\right)\frac{\delta t}{\delta t}\right) = \frac{\partial u}{\partial y} \quad \text{(clockwise)}$$

**Rotation:**

$\omega_z$ (rotation of the element about the z-axis) = average of $\omega_{OB}$ & $\omega_{OA}$

$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad \text{counter clockwise}$$

Similarly,

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

& $$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

where $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

note: $\vec{\omega} = \frac{1}{2} \nabla \times \vec{V} = \frac{1}{2} \vec{\zeta}$

vorticity

$$= \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

For irrotational flows, $\vec{\zeta} = \vec{\omega} = 0$.

Review Example 3.3 on Free + Forced Vortices

<u>Shear</u>:

shear (angular deformation) $= \delta\gamma = \delta\alpha + \delta\beta$

$$\underset{\text{(rate of angular deformation)}}{\dot{\gamma}} = \lim_{\delta t \to 0} \left( \frac{\delta \gamma}{\delta t} \right) = \lim_{\delta t \to 0} \left( \frac{\delta\alpha + \delta\beta}{\delta t} \right)$$

$$= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

For rigid body rotation, $\dot{\gamma} = 0$.

## § Conservation of Mass : The Continuity Equation

Derivation via Reynolds Transport Th$^{\underline{m}}$ (RTT) :

From previous results, the R.T.T. states that

$$\frac{D}{Dt}\int_{sys} \rho \, d\forall = \frac{\partial}{\partial t}\int_{C.\forall} \rho \, d\forall + \int_{C.S.} \rho \vec{V}\cdot\vec{n}\, dA = 0 \qquad (1)$$

( with $b = 1$ in the R.T.T. )

From vector calculus, ( Divergence Th$^{\underline{m}}$ or Gauss Th$^{\underline{m}}$ )

$$\int_{C.S} \rho \vec{V}\cdot\vec{n}\, dA = \int_{C.\forall} \nabla\cdot(\rho\vec{V})\, d\forall$$

Eq$^{\underline{n}}$ (1) can be written as

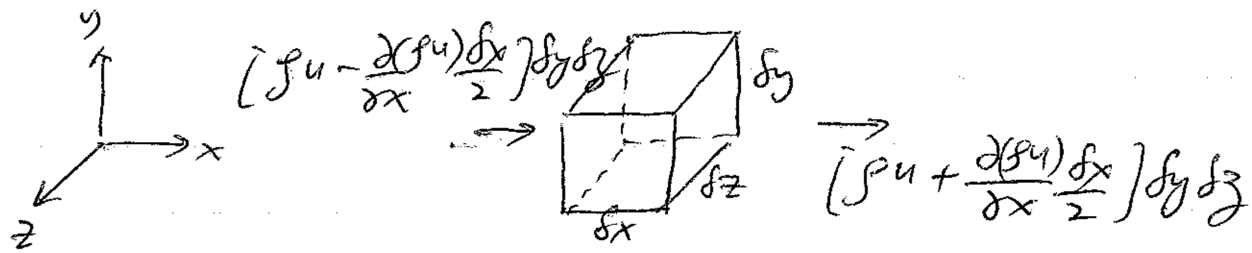
$$\int_{C.\forall} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] d\forall = 0$$

Since it is valid for any arbitrary volume, the integrand must be identically zero, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

or $$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

Derivation via control volume approach




mass flow rate out $\qquad \rho u\Big|_{x+\frac{\delta x}{2}} = \rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$

mass flow rate in $\qquad \rho u\Big|_{x-\frac{\delta x}{2}} = \rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$

$\Longrightarrow$ Net rate of mass outflow in x-dir$^{n}$ $= \left[\rho u\Big|_{x+\frac{\delta x}{2}} - \rho u\Big|_{x-\frac{\delta x}{2}}\right]\delta y\, \delta z$

$$= \frac{\partial(\rho u)}{\partial x}\delta x \cdot \delta y \cdot \delta z$$

Similarly,

Net rate of mass outflow in y-dir$^{n}$ $= \frac{\partial(\rho v)}{\partial y}\delta x\, \delta y\, \delta z$

&

Net rate of mass outflow in z-dir$^{n}$ $= \frac{\partial(\rho w)}{\partial z}\delta x\, \delta y\, \delta z$

Therefore,

Total Net rate of outflow $= \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x \delta y \delta z$

The rate of increase of mass accumulation in the C.V. $= \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$

For mass conservation,

Rate of increase of mass accumulation + Rate of outflow $= 0$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

or $$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

Notes:

(i) For incompressible flow; $\rho$ = constant, the continuity equation becomes:

$$\boxed{\nabla \cdot \vec{V} = 0}$$

(ii) The continuity Equation in polar, cylindrical coordinates is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

## Stream Function for 2-D Flows

For 2-D plane flows, the continuity equation becomes:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

This suggests that if we ~~define~~ introduce a ~~scalar~~ scalar function $\psi$, such that

$$u=\frac{\partial \psi}{\partial y} \quad \& \quad v=-\frac{\partial \psi}{\partial x} \qquad (2)$$

The continuity equation is satisfied identically.

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)\\ \frac{\partial v}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)\end{aligned}\right\}\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

* It can be shown that lines along which $\psi$ is constant are streamlines.

Recall, definition of streamlines are lines that everywhere tangent to the velocities, i.e.

$$\frac{dy}{dx}=\frac{v}{u} \qquad (3)$$

For $\psi = c$

$$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=0$$

$$\Rightarrow d\psi = -v\,dx + u\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{v}{u}$$ same as Eq (3) for a streamline

In cylindrical coordinates;

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad ; \quad v_\theta = -\frac{\partial \psi}{\partial r} \qquad (4)$$

Example :

$$\left.\begin{aligned} u &= 2y \\ v &= 4x \end{aligned}\right\}$$ Find & plot $\psi$'s

Sol$^n$:

(i) $$u = \frac{\partial \psi}{\partial y} = 2y \Rightarrow \psi = y^2 + f_1(x) \qquad (5)$$

(ii) $$v = -\frac{\partial \psi}{\partial x} = 4x \Rightarrow \psi = -2x^2 + f_2(y) \qquad (6)$$

Comparing (5) & (6) gives $f_1(x) = -2x^2$ & $f_2(y) = y^2$

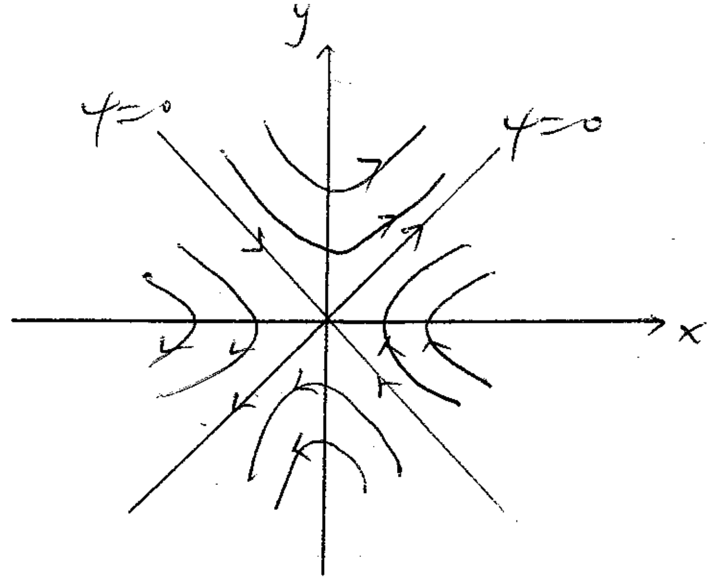
which yields

$$\psi = y^2 - 2x^2$$

$$\psi = c \Rightarrow y^2 - 2x^2 = c \qquad (7)$$

Eq$^n$ of a hyperbola

(see left)




## § Conservation of Linear Momentum (Equations of Motion)

Consider an infinitesimal cubical element of fluid. (Fig 6.4)

**Surface Forces:**

Summing all forces in the x-dirn:

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z \qquad (1)$$

**Body Forces:**

$$\delta F_{bx} = \delta m\, g_x \qquad (2)$$

$g_x$: gravity in x-dirn component

**Force Balance:**

$$\delta F_{sx} + \delta F_{bx} = \delta m \cdot a_x \qquad (3)$$

$$a_x = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z + \delta m\, g_x = \delta m\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x+\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

This is known as the "Cauchy's Eq^N" (4)

Similarly, the above procedure can be repeated in the y and z direction and gives:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right) \quad (5)$$

&

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z+\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right) \quad (6)$$

## § Inviscid Flow

For inviscid flow, $\tau$ are zero, and $\sigma = P$. Eqs (4), (5) & (6) (the Cauchy's Eq^n) become:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x} \quad (4a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y} \quad (5a)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z} \quad (6a)$$

These equations are referred to as "Euler's Equation"

They can be written in vector form as:

$$\rho\left[\frac{\partial \vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right]=\rho\vec{g}-\nabla p \qquad (7)$$

Bernoulli's Equation revisited :

~~It can be shown that,~~ the Euler's Equation, Eq (7), can be used to derive the Bernoulli's Equation. It can be shown that (Details see textbook for it)

$$\frac{p}{\rho}+\frac{V^2}{2}+gz = \text{constant} \quad \text{along a streamline.} \qquad (8)$$

~~for~~: ~~provided~~: inviscid flow, steady flow, incompressible flow } with restrictions as before

Furthermore, it can be shown that, for irrotational flows (i.e. $\vec{\zeta}=\vec{w}=0$), Eq (8) is valid for the whole flow field, i.e.

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2 \qquad (9)$$

is valid between any two points on the flow field.

§ Viscous Flow (Incompressible Newtonian Fluids)

Stress-Deformation relations for incompressible Newtonian fluids are given by:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \qquad (1)$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \qquad (2)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \qquad (3)$$

&

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (4)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (5)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad (6)$$

in Cartesian coordinates and by:

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \qquad (7)$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad (8)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} \qquad (9)$$

&

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \qquad (10)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \qquad (11)$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad (12)$$

Navier-Stokes Eqn :

Substituting Eqns (1) – (6) into the Cauchy's Eqn gives:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\nabla^2 v\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\nabla^2 w\right)$$

§ Some Simple Solutions for Viscous Incompressible Fluids

(1) "Steady" Laminar Flow b/w Fixed // Plates :


Steady flow, fully developed (does not change in x)

$u = u(y)$

$v = 0$

The N-S equations become : $w = 0$

x : $$\rho\Big(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0}+u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\substack{0\\u=u(y)}}+\underbrace{\cancel{v}}_{0}\frac{\partial u}{\partial y}+\underbrace{\cancel{w}}_{0}\frac{\partial u}{\partial z}\Big)=-\frac{\partial p}{\partial x}\underbrace{\cancel{\rho g_x}}_{0}+\mu\Big(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0}+\frac{\partial^2 u}{\partial y^2}+\underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0}\Big)$$

$$\Rightarrow \qquad 0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$y$: $\rho\underbrace{\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)}_{\substack{\shortparallel \\ 0 \text{ since } v=0}} = -\frac{\partial p}{\partial y}+\rho g_y+\underbrace{\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)}_{\substack{\shortparallel \\ 0 \\ \text{since } v=0}}$

$$\Rightarrow \quad 0 = -\frac{\partial p}{\partial y} - \rho g \qquad (2)$$

$z$: $\rho\underbrace{\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)}_{\substack{\shortparallel \\ 0 \text{ since } w=0}} = -\frac{\partial p}{\partial z}+\rho \underbrace{g_z}_{\substack{\shortparallel \\ 0}}+\mu \underbrace{\nabla^2 w}_{\substack{\shortparallel \\ 0}}$

$$\Rightarrow \quad 0 = \frac{\partial p}{\partial z} \qquad (3)$$

Eqs (2) & (3) gives

$$p = -\rho g y + f_1(x) \qquad (4)$$

$$\Rightarrow \frac{\partial p}{\partial x} = f_1'(x) \quad \text{; a function of } x \text{, at the most}$$

From Eq (1)

$$\cancel{\frac{\partial^2 u}{\partial x^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}}$$

~~Since $u=u(y)$ & $\frac{\partial p}{\partial x}=f(x)$, it can be written as~~

$$\cancel{\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}}$$

From Eq. (1)

$$\underbrace{\frac{d^2u}{dy^2}}_{f(y)} = \frac{1}{\mu}\underbrace{\frac{\partial p}{\partial x}}_{f(x)} = \text{constant} \qquad (5)$$

Note: Since $\frac{d^2u}{dy^2}$ is a function of $y$ and $\frac{\partial p}{\partial x}$ is a function of $x$. They can only be equal to a constant independent of $x$ and $y$.

Integrating Eq (5) gives:

$$\frac{du}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)y + C_1 \qquad (6)$$

Integrating again gives:

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + C_1 y + C_2 \qquad (7)$$

$C_1$ & $C_2$ can be determined by the boundary conditions as follow:

(i) $u = 0$ at $y = \pm h$ (no slip condition)

Eq (7) gives $C_1 = 0$

$$C_2 = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$

Eq (7) becomes:

$$\boxed{u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)} \qquad (8)$$

Some useful results:

(i) Flow rate: $\dot{V}$

$$\dot{V} = \int_{-h}^{h} u\,dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)\,dy$$

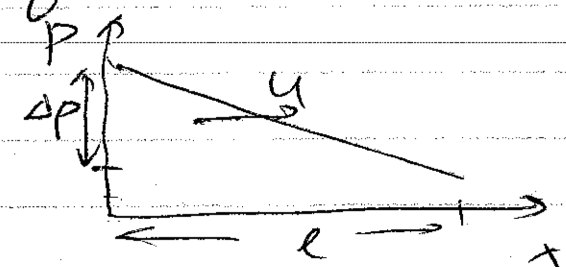
$$= \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{y^3}{3} - h^2 y\right]_{-h}^{h}$$

(per unit depth)

$$= \boxed{-\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)} \qquad (9)$$

Note, $\frac{\partial p}{\partial x}$ is a constant, i.e. constant pressure gradient in the x direction and is negative for flow in +ve x dirn

$$\therefore -\frac{\partial p}{\partial x} = \frac{\Delta p}{l}$$




Eq (9) can be written as

$$\boxed{\dot{V} = \frac{2h^3 \Delta p}{3\mu l}} \qquad (10)$$

(*) It can be shown that the addition of gravity will result only in the addition of hydrostatic pressure on the pressure term. ~~the~~

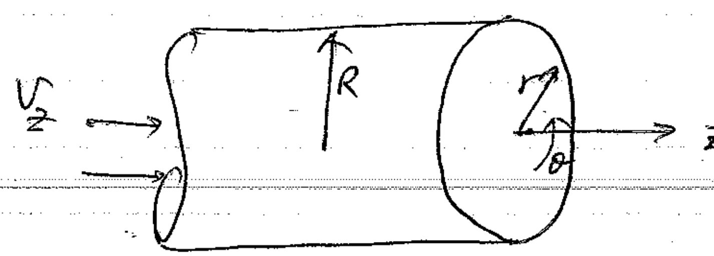
(ii) $\bar{u}$ : mean velocity

$$\bar{u} = \frac{\dot{V}}{2h} = \frac{h^2 \Delta p}{3\mu l} \quad \left( = -\frac{h^2}{3\mu}\left(\frac{\partial p}{\partial x}\right) \right) \tag{11}$$

(iii) $u_{max}$ : maximum velocity

$$u_{max} = \frac{-h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{h^2 \Delta p}{2\mu l} = \frac{3}{2}\bar{u} \tag{12}$$

(II) 1. <u>Steady Laminar Flow in Circular Tubes</u> (Poiseuille Flow)




$$\begin{cases} V_z = V_z(r) \\ V_r = 0 \\ V_\theta = 0 \end{cases}$$

Consider the flow thru a circular tube of radius R. Cylindrical coordinates system $(r, \theta, z)$ is the obvious choice with z along the the center of the pipe. Let's neglect ~~the~~ gravity for the time being. (*) The N-S equations become:

$$0 = -\frac{\partial p}{\partial r} \tag{1}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] \qquad (3)$$

(note "$\partial$" has been changed to "$d$" since $v_z = v_z(r)$ only)

Equations (1) + (2) $\Rightarrow$ $p = p(z)$ at the most

$$\Rightarrow \underbrace{\frac{\partial p}{\partial z}}_{= \frac{dp}{dz}} = f(z)$$

Eq (3) becomes

$$\frac{dp}{dz} = \underbrace{\frac{\partial p}{\partial z}}_{f(z)} = \underbrace{\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)}_{f(r)} = \text{constant} \qquad (4)$$

Again, as in the previous case, the two terms must be equal to a constant, independent of $z$ and $r$. ~~Integrating Eq (4) once gives:~~

$$\cancel{\mu r \frac{\partial p}{\partial z} = \frac{d}{dr}\left(r\frac{dv_z}{dr}\right)}$$

$$\cancel{\Rightarrow \mu\frac{r^2}{2}\frac{\partial p}{\partial z} + c_1 = r\frac{dv_z}{dr}} \qquad \cancel{(5)}$$

~~Integrating again gives:~~

From Equation (*)

$$\frac{r}{\mu}\frac{dp}{dz} = \frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

$$\Rightarrow \quad r\frac{dv_z}{dr} = \int \frac{r}{\mu}\frac{dp}{dz}\,dr + C_1$$

$$= \frac{r^2}{2\mu}\frac{dp}{dz} + C_1$$

$$\Rightarrow \quad \frac{dv_z}{dr} = \frac{r}{2\mu}\frac{dp}{dz} + C_1/r$$

$$\Rightarrow \quad v_z = \frac{r^2}{4\mu}\frac{dp}{dz} + C_1 \ln r + C_2 \qquad (5)$$

B.C. (i) $v_z$ is finite
(ii) $v_z = 0$ at $r = R$

(i) gives $C_1 = 0$

(ii) gives $0 = \frac{R^2}{4\mu}\frac{dp}{dz} + C_2$

Eq (5) becomes

$$\boxed{v_z = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - R^2)} \qquad (6)$$

Some useful results:

(i) Flow rate; $\dot{V}$

$$\dot{V} = \int_0^R v_z\, 2\pi r\,dr$$

$$= \int_0^R \frac{1}{4\mu}\frac{dp}{dz}(r^2 - R^2)\, 2\pi r\, dr$$

$$= \boxed{-\frac{\pi R^4}{8\mu}\frac{dp}{dz}} \qquad (7)$$

Again as before, $-\frac{dp}{dz} = \frac{\Delta p}{\ell}$, Eq (7) becomes

$$\boxed{\dot{V} = \frac{\pi R^4 \Delta p}{8\mu \ell}} \qquad (8)$$

(ii) mean velocity, $\bar{v}_z$

$$\boxed{\bar{v}_z = \frac{\dot{V}}{\pi R^2} = \frac{R^2 \Delta p}{8\mu \ell}} \qquad (9)$$

(iii) Maximum velocity, $v_{max}$

$$v_{max} = v_z\Big|_{r=0} = -\frac{R^2}{4\mu}\frac{dp}{dz} = \frac{R^2 \Delta p}{4\mu \ell}$$

$$= 2\bar{v}_z \qquad (10)$$

Note: Eq (6) can be written as:

$$\boxed{\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2} \qquad (11)$$