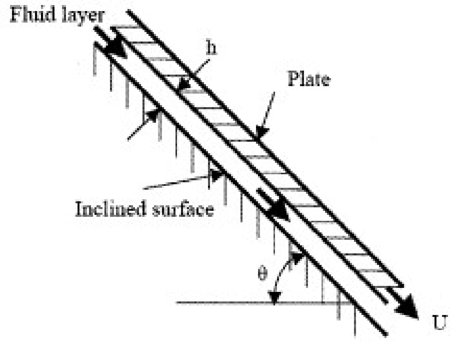
Fluid Mechanics
(Final Exam)

Name: Id:

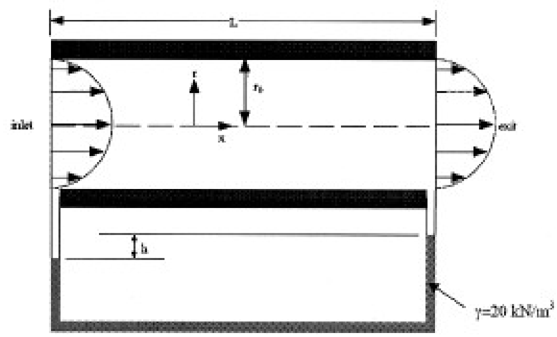
1. (10 points) A 1:30 scale model of a ship is to be tested in a towing tank. Determine the required kinematic viscosity of the model fluid so that both the Reynolds number and the Froude number are the same for model and prototype. The prototype fluid is to be seawater at 60oF.
2. (20 points) For a steady, two-dimensional incompressible flow in x-y plane, show that the z component of vorticity ξand the stream function are related by the equationξ.

NOT COVERED IN CLASS

3. (20 points) A layer of incompressible fluid with a uniform thickness, h, between a large plate and an inclined surface moves steadily down parallel to the inclined surface, as shown in the figure. The plate moves with a constant velocity while the inclined surface is fixed at an angle,. The plate has a surface area of A and a mass of M. Assume the fluid has a density of and a constant viscosity of μ; the flow is steady and laminar. The acceleration of gravity, g, is a constant. Starting with the full Navier-Stokes equations, derive an expression for the velocity of the plate, U, in terms of μ, ρ, θ, A, M, g, and h.



4. (30 points) A turbulent Newtonian flow of liquid water through a horizontal section of pipe is shown in the figure below. The pipe has a circular cross-section with constant radius, r_0. The time-averaged fully developed velocity profile is approximated as $u(r)=u_{max}\left(1-\frac{r^4}{r_0^4}\right)$.



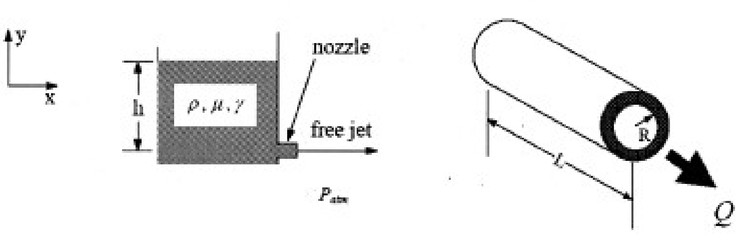
Assume a steady flow with $u_{max}=100.0\ m/s$, $L=500.0\ m$, and $r_0=0.25\ m$.

a. Calculate the mass flow rate through the pipe.
b. Calculate the differential height reading, h, on the manometre.
c. Calculate the reaction force in the x-direction required to hold the pipe in place.
d. If the flow is adiabatic, calculate the temperature change between the inlet and the exit?
e. Calculate the Reynolds number. Assume the reference length is r_0 and the reference velocity is the average cross-section velocity.
f. If the water is replaced with an inviscid fluid with the same volumetric flow rate, how would the answers to parts b.-e. change?

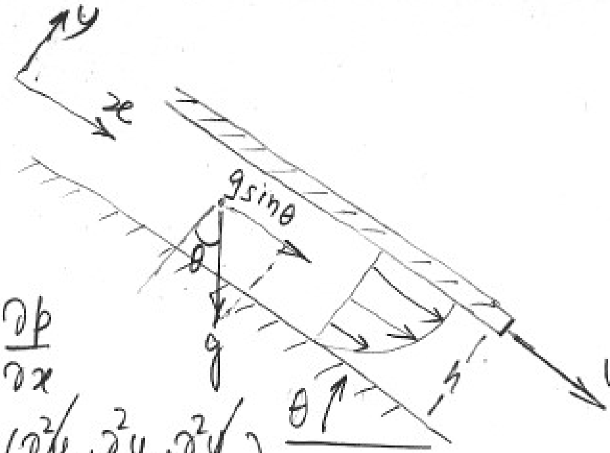
5. (20 points) A tank with the nozzle shown below is open to the atmosphere. The nozzle has a length of L and constant radius of R. The liquid exits the nozzle as a free jet into the atmosphere. The desired volume flow rate for the nozzle is Q. Find an expression for the required height of the liquid in the tank, h, in terms of the given variables under the following assumptions:

a. inviscid flow in the nozzle
b. laminar Newtonian viscous flow in the nozzle

Assume the tank is large enough that the liquid height does not change with time. Also assume the flow in the nozzle is steady and incompressible. Acceleration of gravity is g.



3.



Choose x-axis along the incline:

NS eqn. in x-dir:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0,\ \text{steady}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\substack{0\\ \text{fully dev.}}} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{0,\ v=0} + \underbrace{\cancel{w\frac{\partial u}{\partial z}}}_{0,\ \frac{\partial u}{\partial z}=w=0}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) + \rho g_x$$

$$\Rightarrow \quad 0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \rho g \sin\theta$$

Since thickness is uniform, h = constt. therefore $\frac{\partial p}{\partial x} = 0$

So we have: $\mu\frac{\partial^2 u}{\partial y^2} = -\rho g \sin\theta$

Integrating twice gives:

$$u = -\frac{\rho g}{2\mu}\sin\theta\, y^2 + C_1 y + C_2 \quad \text{——— (I)}$$

Boundary Conditions are:

$y = 0 \qquad u = 0$

at $\; y = h \qquad u = U$

First B.C. gives $\quad C_2 = 0$

Second B.C gives $\quad C_1 = \frac{U}{h} + \frac{\rho g h}{2\mu}\sin\theta$

$\Big\}$ substitute in (I)

$$\therefore \quad u(y) = y\left[\frac{U}{h} + \frac{\rho g}{2\mu}\sin\theta\,(h-y)\right] \quad \text{——— (II)}$$

Since plate is moving with constt velocity ($\Rightarrow$ Eqvm.) sine component of its weight is balanced by the shear force τA.

i.e. $Mg\sin\theta = \tau A$

$$\tau = \mu \frac{\partial u}{\partial y}\Big|_{y=h} = \frac{Mg\sin\theta}{A} \qquad \text{(III)}$$

$$\frac{\partial u}{\partial y} = \frac{U}{h} + \frac{\rho g h \sin\theta}{2\mu} - \frac{\rho g y \sin\theta}{\mu}$$

$$\frac{\partial u}{\partial y}\Big|_{y=h} = \frac{U}{h} + \frac{\rho g h \sin\theta}{2\mu} - \frac{\rho g h \sin\theta}{\mu} = \frac{U}{h} - \frac{\rho g h}{2\mu}\sin\theta$$

$$\tau = \mu \frac{\partial u}{\partial y}\Big|_{y=h} = \mu\left(\frac{U}{h} - \frac{\rho g h \sin\theta}{2\mu}\right)$$

Substituting this in (III)

$$\mu\left(\frac{U}{h} - \frac{\rho g h \sin\theta}{2\mu}\right) = \frac{Mg\sin\theta}{A}$$

$$\mu\frac{U}{h} = \frac{Mg\sin\theta}{A} + \frac{\rho g h \sin\theta}{2}$$

$$U = \frac{h}{\mu}\left[\frac{Mg\sin\theta}{A} + \frac{\rho g h \sin\theta}{2}\right]$$

so $$U = \frac{g h \sin\theta}{\mu}\left(\frac{M}{A} + \frac{\rho h}{2}\right)$$ Ans

#4.

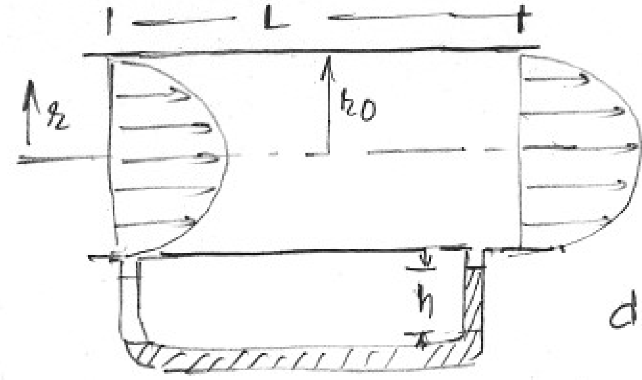
$\rho = 999\ kg/m^3$

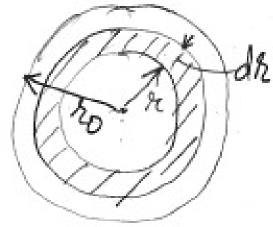
$\gamma = 9.8\ kN/m^3$

$\mu = 1.12\times10^{-3}\ Ns/m^2$

$c_v = 4.187\ kJ/kg\,K.$

$u(r) = u_{max}\left(1-\left(\frac{r}{r_0}\right)^4\right)$





$dA = 2\pi r\,dr$

ⓐ $\dot{m} = \int \rho u_r \cdot dA = \int_0^{r_0} \rho u_{max}\left(1-\frac{r^4}{r_0^4}\right) 2\pi r\,dr$

$= \rho u_{max} 2\pi \int_0^{r_0} \left(r - \frac{r^5}{r_0^4}\right) dr$

$= 2\pi \rho u_{max}\left[\frac{r^2}{2} - \frac{r^6}{6r_0^4}\right]_0^{r_0} = 2\pi\rho u_{max}\left[\frac{r_0^2}{2} - \frac{r_0^6}{6r_0^4}\right]$

$\dot{m} = \frac{2}{3}\pi\rho u_{max} r_0^2 = \frac{2}{3}\times\pi\times999\times100\times0.25^2$

$\Rightarrow \quad \dot{m} = 13077\ kg/s \quad$ Ans.

ⓑ Manometer gives:

$p_i + \gamma_w h = p_e + \gamma_m h.$

$\Rightarrow \quad p_i - p_e = (\gamma_M - \gamma_w) h$

$h = \frac{p_i - p_e}{\gamma_M - \gamma_w} \quad \text{———} \quad (\mathrm{I})$

Apply momentum eqn:

$\int(\rho V\cdot dA)V = -\int p\,dA + F_{visc.} = -\left[-p_i A + p_e A\right] - \tau 2\pi r_0 L$

(0, constt area dev. flow)

$0 = (p_i - p_e)\pi r_0^2 - \tau\, 2\pi r_0 L.$

$$-(p_i - p_e)\pi r_0^2 = -\tau 2\pi r_0 L$$

$$p_i - p_e = \frac{\cancel{-}\tau 2\cancel{\pi} r_0 L}{\cancel{-}\cancel{\pi} r_0^2} = \frac{2\tau L}{r_0}$$

and $\tau = \mu \left.\frac{du}{dr}\right|_{r=r_0}$

$$= \mu \left| \frac{u_{max} \cdot 4r^3}{r_0^4} \right|_{r=r_0} = \frac{4\mu u_{max}}{r_0}$$

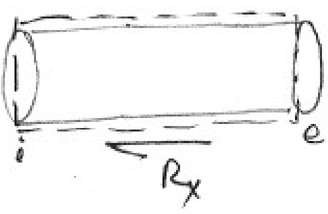
$$\therefore \quad p_i - p_e = \frac{2\tau L}{r_0} = \frac{8\mu u_{max} L}{r_0^2} = \frac{8 \times 1.12 \times 10^{-3} \times 100 \times 500}{0.25^2}$$

$$p_i - p_e = 7168$$

Substitute this in. (I)

$$h = \frac{p_i - p_e}{\gamma_M - \gamma_W} = \frac{7168}{20\times10^3 - 9.8\times10^3} = 0.703 \text{ m} \quad \text{Ans}$$

(c) momentum eqn:



$$\cancel{\int_{cs}(\rho V \cdot n\, dA)V}_{\,0} = -\int p\,dA - R_x$$

$$= -\left[-p_i + p_e\right]\pi r_0^2 - R_x$$

$$\Rightarrow R_x = (p_i - p_e)\pi r_0^2 = 7168 \times \pi \times 0.25^2 = 1407 \text{ N}$$

Ans

(d) Energy Eqn:

$$\check{u}_i - \check{u}_e + \frac{p_i - p_e}{\rho} + \cancel{\frac{V_i^2 - V_e^2}{2}}_{\,0} + g\cancel{(z_e - z_i)}_{\,0} = \cancel{q}_{\,0}$$

$$\Rightarrow \check{u}_i - \check{u}_e = \frac{p_e - p_i}{\rho}$$

$$\check{u}_e - \check{u}_i = \frac{p_i - p_e}{\rho}$$

$$c_v(T_e - T_i) = \frac{p_i - p_e}{\rho}$$

$$c_v\, dT = \frac{7168}{999} = 7.175$$

$$dT = 7.175 / 4.187 \times 10^3 = 1.714 \times 10^{-3}\, K. \quad \underline{Ans}$$

(e) $$Re = \frac{\rho V L}{\mu} = \frac{\rho\, \dot{m} \cdot r_0}{\rho A \cdot \mu} = \frac{\dot{m} r_0}{\mu A}$$

$$= \frac{13077 \times 0.25}{1.12 \times 10^{-3} \times \pi \times 0.25^{2}} = 14.8 \times 10^6 \quad \underline{Ans}$$

(f) inviscid means $\mu = 0$

(a) is unaffected

(b) since $h = \dfrac{p_i - p_e}{\gamma_M - \gamma_W}$ and $p_i - p_e = \dfrac{8 \cancel{\mu} u_{max} L}{r_0^2}$

$\therefore \quad h = 0$

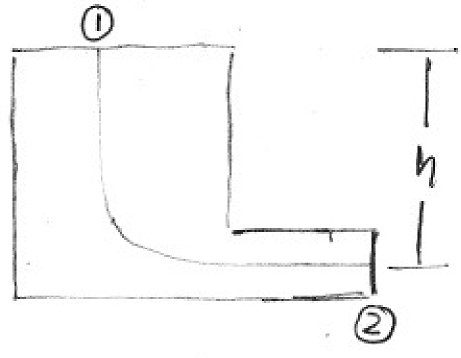
(c) similiary since $R_x = (\cancel{p_i - p_e})\pi r_0^2 = 0$
$\qquad\qquad\qquad\qquad 0$

(d) $$\Delta T = \frac{\cancel{p_i - p_e}}{c_v \rho} = 0$$
$\qquad 0$

(e) $$Re = \frac{\rho V L}{\mu} = \infty.$$

#5 (a) Inviscid flow throughout

Bernoulli eqn.



$$\cancel{p_1} + \frac{1}{2}\rho \cancel{V_1^2} + \gamma z_1 = \cancel{p_2} + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

$$\gamma(z_1 - z_2) = \frac{1}{2}\rho V_2^2$$

$$\gamma h = \frac{1}{2}\rho V_2^2 \quad \text{—— (I)}$$

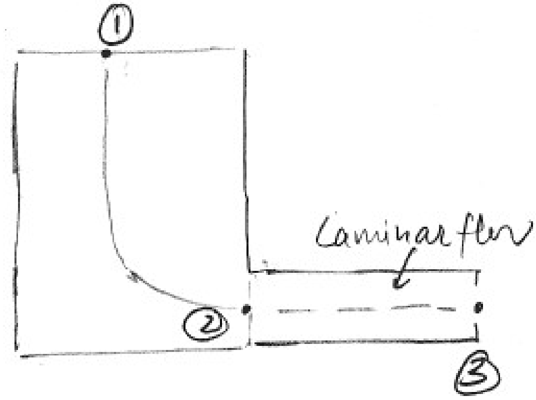
$$V_2 = Q/\pi R^2$$

Therefore $h = \frac{1}{2\gamma}\rho\left(\frac{Q}{\pi R^2}\right)^2 = \frac{\rho Q^2}{2\gamma\pi^2 R^4}$ Ans

(b) Flow in the nozzle is laminar ⇒ Viscous

therefore we can't apply Bernoulli eqn between (2) and (3)

So Inside the nozzle we apply Poiseulle's relation:



$$Q = \frac{\pi R^4 \Delta p}{8\mu L} = \frac{\pi R^4 (p_2 - p_3)}{8\mu L} = \frac{\pi R^4}{8\mu L}(p_2 - p_{atm}) \quad \text{—— (II)}$$

Apply Bernoulli's relation between (1) and (2)

$$p_1 + \frac{1}{2}\rho \cancel{V_1^2} + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

$$p_{atm} + \gamma(z_1 - z_2) - \frac{1}{2}\rho V_2^2 = p_2.$$

$$\Rightarrow \quad p_2 = p_{atm} + \gamma h - \frac{1}{2}\rho V_2^2 \qquad \text{sub. this in (II)}$$

$$Q = \frac{\pi R^4}{8\mu L}\left[\cancel{p_{atm}} + \gamma h - \frac{1}{2}\rho V_2^2 - \cancel{p_{atm}}\right] = \frac{\pi R^4}{8\mu L}\left[\gamma h - \frac{1}{2}\rho V_2^2\right]$$

$$\Rightarrow \quad 8\mu L Q = \pi R^4\left(\gamma h - \frac{1}{2}\rho V_2^2\right)$$

$$\Rightarrow \quad h = \frac{1}{\gamma}\left[\frac{8\mu L Q}{\pi R^4} + \frac{\rho Q^2}{2\pi^2 R^4}\right] = \frac{8\mu L Q}{\gamma\pi R^4} + \frac{Q^2}{2g\pi^2 R^4} \quad \text{Ans}$$