#1. Consider the following steady, two dimensional velocity field:

$$\boldsymbol{V} = (u, v) = (0.5 + 1.2\,x)\boldsymbol{i} + (-2.0 - 1.2\,y)\boldsymbol{j}$$

Is there a stagnation point in this flow field? If so where is it?

Solve: At stagnation point

$u = 0 \Rightarrow 0.5 + 1.2x = 0 \Rightarrow x = -0.5/1.2 = -0.417$

$v = 0 \Rightarrow -2.0 - 1.2y = 0 \Rightarrow y = -1.67$

So, the stagnation point is $(-0.417, -1.67)$

#2. A steady, incompressible, two dimensional velocity field is given by the following components in the x-y plane:

$$u = 1.1 + 2.8\,x + 0.65\,y, \; v = 0.98 - 2.1\,x - 2.8\,y$$

Calculate the acceleration field and calculate the acceleration the fluid experiences at point $(x, y) = (-2, 3)$.

Solve: $a = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V} = u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y}$

$$u\frac{\partial V}{\partial x} = u\frac{\partial}{\partial x}\left[(1.1+2.8x+0.65y)\vec{i} + (0.98-2.1x-2.8y)\vec{j}\right]$$
$$= u(2.8\vec{i} - 2.1\vec{j})$$

$$v\frac{\partial V}{\partial y} = v\frac{\partial}{\partial x}\left[(1.1+2.8x+0.65y)\vec{i} + (0.98-2.1x-2.8y)\vec{j}\right]$$
$$= v(0.65\vec{i} - 2.8\vec{j})$$

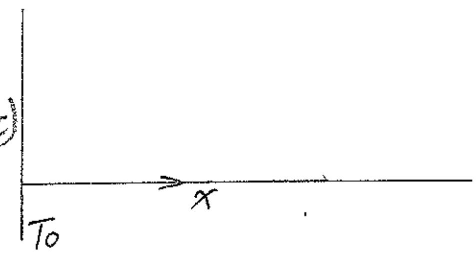
So, $a = (1.1+2.8x+0.65y)(2.8\vec{i}-2.1\vec{j}) + (0.98-2.1x-2.8y)(0.65\vec{i} - 2.8\vec{j})$
$$= (3.717 + 6.475x)\vec{i} - (5.054 - 6.495y)\vec{j}$$

At point $(x, y) = (-2, 3)$

$$a = [3.717 + (6.475)(-2)]\vec{i} - [5.054 - (6.495)(3)]\vec{j}$$
$$= -9.233\vec{i} + 14.37\vec{j}$$

#3. The temperature T, in a long tunnel is known to vary approximately as $T = T_0 - \alpha e^{-x/L} \sin(2\pi t/\tau)$, where T_0, α, L and τ are constants, and x is measured from the entrance. A fluid moves into the tunnel with a constant speed, U. Obtain an expression for the rate of change of temperature experienced by the fluid.

Solve: Given: $T = T_0 - \alpha e^{-x/L} \sin(2\pi t/\tau)$



$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (V \cdot \nabla) T$$

$$\frac{\partial T}{\partial t} = 0 - \alpha e^{-x/L} \frac{2\pi}{\tau} \cos\left(\frac{2\pi t}{\tau}\right) = -\frac{2\pi\alpha}{\tau} e^{-x/L} \cos\left(\frac{2\pi t}{\tau}\right)$$

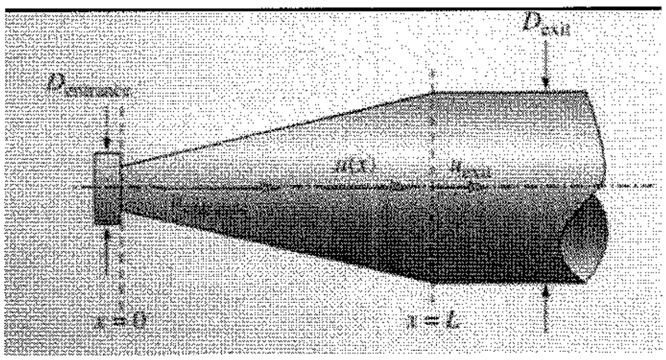
$$(V \cdot \nabla) T = u \frac{\partial T}{\partial x} = u \frac{\partial}{\partial x}\left[T_0 - \alpha e^{-x/L} \sin(2\pi t/\tau)\right]$$

$$= \frac{u\alpha}{L} e^{-x/L} \sin(2\pi t/\tau)$$

So, $$\frac{DT}{Dt} = -\frac{2\pi\alpha}{\tau} e^{-x/L} \cos\left(\frac{2\pi t}{\tau}\right) + \frac{u\alpha}{L} e^{-x/L} \sin(2\pi t/\tau)$$

$$= \alpha e^{-x/L}\left[\frac{u}{L} \sin\left(\frac{2\pi t}{\tau}\right) - \frac{2\pi}{L} \cos\left(\frac{2\pi t}{\tau}\right)\right]$$

#4. Consider steady flow of air through the diffuser of a wind tunnel as shown. Along the centerline of the diffuser, the air speed decreases from $u_{entrance}$ to u_{exit} as sketched. Measurements reveal that the centerline airspeed decreases parabolically through the diffuser. Write an equation for the centerline speed *u(x)* and the fluid acceleration *a(x)* along the centerline based on the parameters given, $x = 0$ to $x = L$. For $L = 2.0\ m$, $u_{entrance} = 30\ m/s$ and $u_{exit} = 5.0\ m/s$, calculate the acceleration at $x = 0$ and $x = 1.0\ m$.



Solve: According to centerline airspeed decrease parabolically,

$$u(x) = u_{entrance} - a x^2$$

At $x = 0$, $u = u_{entrance} = 30\ m/s$

$x = L = 2.0\ m$, $u = u_{exit} = 5.0\ m/s$

$\therefore\ 5 = 30 - aL^2 = 30 - 4a \Rightarrow a = \frac{25}{4} = 6.25$

So $u(x) = 30 - 6.25x^2$

$$a = u\frac{\partial V}{\partial x} = (30 - 6.25x^2)(-12.5x)$$

At $x = 0$, $a = 0$

$x = 1.0\ m$, $a = (30 - 6.25)(-12.5)$

$= -296.875\ m/s^2$

#5. After discarding any constants of integration, determine the appropriate value of the unknown velocities w or v that satisfy the equation of three-dimensional incompressible continuity for

(a) $u = x^2yz \quad v = -y^2x$

(b) $u = x^2 + 3z^2x \quad w = -z^3 + y^2$

Solve: three-dimensional incompressible continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

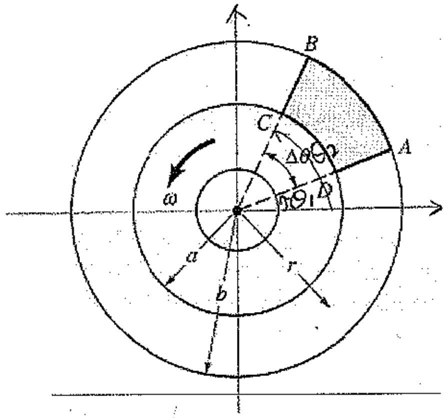
a) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow 2xyz - 2xy + \frac{\partial w}{\partial z} = 0$

$\Rightarrow w = \int (2xyz - 2xy)\,dz = xyz^2 - 2xyz$

b) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow 2x + 3z^2 + \frac{\partial v}{\partial y} - 3z^2 = 0$

$\Rightarrow v = \int -2x\,dy = -2xy$

#6. The forced vortex shown below is composed of streamlines that are concentric circles. The velocity is given by $v_\theta = \omega r$ and $v_r = 0$ where ω is the constant angular velocity of the vortex. Is this flow irrotational? Determine the circulation around path *ABCD*.



Solve:

$$\Gamma = \oint_{ABCD} \vec{V} \cdot d\vec{s} = \int_{AB} V_\theta b \, d\theta + \int_{BC} V_r \, dr + \int_{CD} V_\theta a \, d\theta + \int_{DA} V_r \, dr$$

Since $V_r = 0$, $V_\theta = \omega r$

$$\Gamma = \int_{\theta_1}^{\theta_2} \omega b^2 \, d\theta + \int_{\theta_2}^{\theta_1} \omega a^2 \, d\theta$$

$$= \omega b^2 (\theta_2 - \theta_1) + \omega a^2 (\theta_1 - \theta_2)$$

$$= \omega \Delta\theta (b^2 - a^2)$$

$\therefore$ This flow is not irrotational.

#7. A two-dimensional flow is defined by

$$u = -\frac{Ky}{x^2+y^2}, \qquad v = +\frac{Kx}{x^2+y^2}$$

Where K = constant. Is this flow incompressible and irrotational? If so, find its velocity potential and stream function. Plot a few potential and stream lines, and interpret the flow pattern.

Solve: For incompressible flow, $\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2kxy}{(x^2+y^2)^2} - \frac{2kxy}{(x^2+y^2)^2} = 0$$

$\therefore$ this flow is incompressible.

For irrotational flow, $\nabla\times\vec{V} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

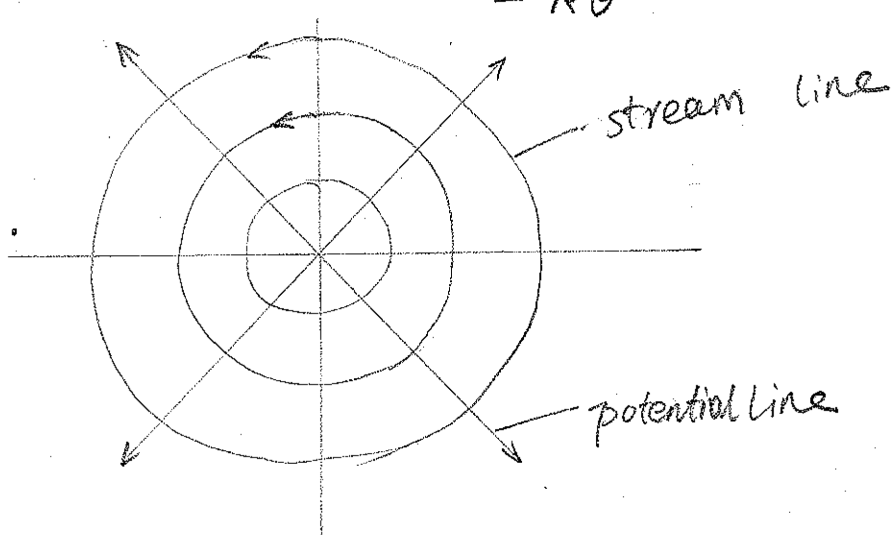
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{2ky^2 - k(x^2+y^2)}{(x^2+y^2)^2} + \frac{2kx^2 - k(x^2+y^2)}{(x^2+y^2)}$$

$$= 0$$

$\therefore$ this flow is irrotational.

$$u = \frac{\partial\phi}{\partial x} = -\frac{ky}{x^2+y^2} \Rightarrow \phi = \int -\frac{ky}{x^2+y^2}dx = -k\tan^{-1}\left(\frac{x}{y}\right)$$

$$= k\theta$$



Flow pattern: source.

#8. A two-dimensional incompressible flow field is defined by the velocity components

$$u = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \qquad v = -2V\frac{y}{L}$$

where V and L are constants. If they exist, find the stream function and velocity potential.

Solve: $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2V}{L} - \frac{2V}{L} = 0$

$\therefore$ satisfy continuity means ψ exist.

$\nabla \times \vec{V} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{2V}{L} - 0 = -\frac{2V}{L} \neq 0$

$\therefore$ not satisfy irrotational means ϕ does not exist.

According to: $\frac{\partial \psi}{\partial y} = u = 2V\left(\frac{x}{L} - \frac{y}{L}\right)$

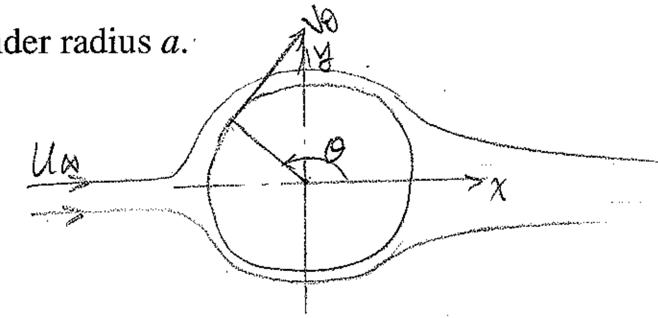
$\Rightarrow \psi = \int 2V\left(\frac{x}{L} - \frac{y}{L}\right) dy = \frac{2Vxy}{L} - \frac{Vy^2}{L} + C(x)$

According to: $\frac{\partial \psi}{\partial x} = -v = 2V\frac{y}{L}$

$\Rightarrow \psi = \int 2V\frac{y}{L} dx = 2V\frac{xy}{L} + C(y)$

So, $\psi = \frac{2Vxy}{L} - \frac{Vy^2}{L} + C$

#9. A circular cylinder is fitted with two surface-mounted pressure sensors, to measure p_a at $\theta=180^o$ and p_b at $\theta=105^o$. The intention is to use the cylinder as a stream velocimeter. Using potential flow theory, derive a formula for estimating U_∞ in terms of p_a, p_b, ρ, and the cylinder radius a.



Solve: Tangential Velocity on the surface of circular cylinder in free stream velocity U_∞ is

$$V_\theta = -2U_\infty \sin\theta$$

Apply Bernoulli equation between free stream and point a $\theta = 180°$,

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_a + \frac{1}{2}\rho U_a^2 = p_a + \frac{1}{2}\rho(-2U_\infty \sin 180°)^2$$

$$\Rightarrow p_\infty = p_a - \frac{1}{2}\rho U_\infty^2 \qquad (1)$$

Apply Bernoulli equation between free stream and point b $\theta = 105°$

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_b + \frac{1}{2}\rho U_b^2 = p_b + \frac{1}{2}\rho(-2U_\infty \sin 105°)^2$$

$$\Rightarrow p_\infty = p_b + 2\rho U_\infty^2 \sin^2(105°) - \frac{1}{2}\rho U_\infty^2 \qquad (2)$$

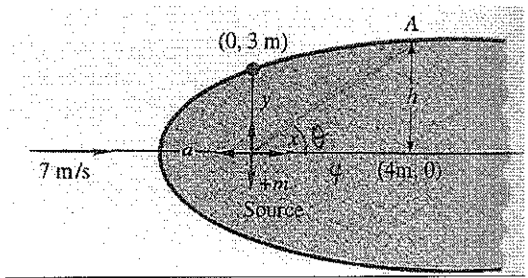
According to equation (1) & (2).

$$p_a - \frac{1}{2}\rho U_\infty^2 = p_b + 2\rho U_\infty^2 \sin^2(105°) - \frac{1}{2}\rho U_\infty^2$$

$$\Rightarrow p_a - p_b = 2\rho U_\infty^2 \sin^2(105°)$$

$$\Rightarrow U_\infty = \sqrt{\frac{p_a - p_b}{2\rho \sin^2(105°)}}$$

#10. A Rankine half-body is formed as shown. For the stream velocity and body dimension shown, compute (a) the source strength m in m^2/s, (b) the distance a, (c) the distance h, and (d) the velocity magnitude at point A.



Solve: for Rankine half body

$$\psi = U r \sin\theta + m\theta \quad , \quad m = \frac{Q}{2\pi}$$

At stagnation point ; $r = a$, $\theta = \pi$

$$\psi = m\pi$$

$$V_r = U = \frac{m}{a} \Rightarrow a = \frac{m}{U}$$

So the equation of streamline passing through the stagnation point is:

$$a\pi U = U r \sin\theta + aU\theta \Longrightarrow r = \frac{a(\pi - \theta)}{\sin\theta}$$

b) At point $(0, 3)$, $r = 3$, $\theta = \frac{\pi}{2}$.

$$3 = \frac{a(\pi - \frac{\pi}{2})}{\sin(\frac{\pi}{2})} = \frac{\pi a}{2} \Rightarrow a = \frac{6}{\pi} = 1.91 \text{ m}.$$

a) $$m = aU = \left(\frac{6}{\pi}\right)(7) = 13.37 \text{ m/s}$$

c) At point $(4, 0)$, $r = \frac{4}{\cos\theta}$

$$\frac{4}{\cos\theta} = \frac{1.9(\pi - \theta)}{\sin\theta} \Rightarrow \theta = 47.8°.$$

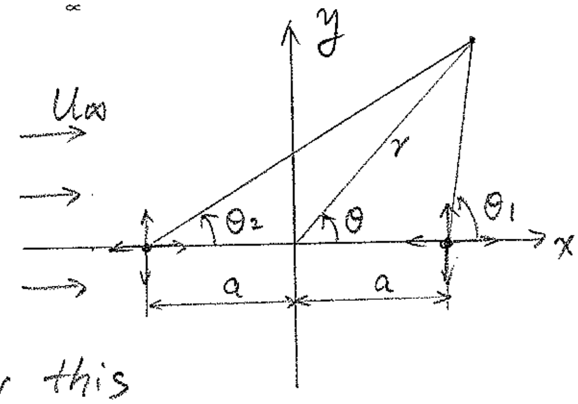
$$h = 4\tan\theta = 4\,tg\,(47.8°) = 4.44 \text{ m}$$

d) $$V^2 = V_r^2 + V_\theta^2 = U^2\left(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)$$

$$= 7^2\left(1 + \frac{(2)(1.91)}{4/(\cos(47.8°))}\cos(47.8°) + \frac{1.91^2}{4^2/\cos(47.8°)^2}\right) = (8.67)^2$$

$\therefore V = 8.67$ m/s

#11. Sketch the streamlines, especially the body shape, due to equal line sources +m at (-a,0) and (+a,0) plus a uniform stream $U_\infty = ma$.

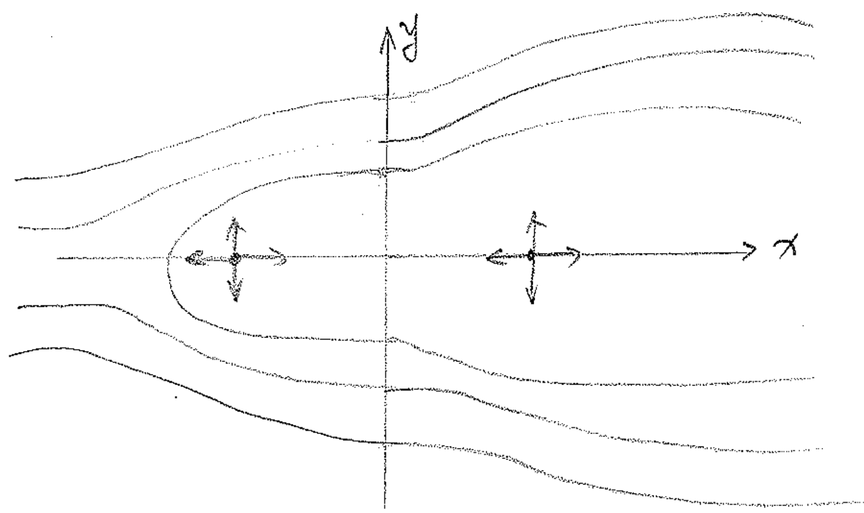


Solve: $\psi = \psi_{uniform} + \psi_{two\ source}$

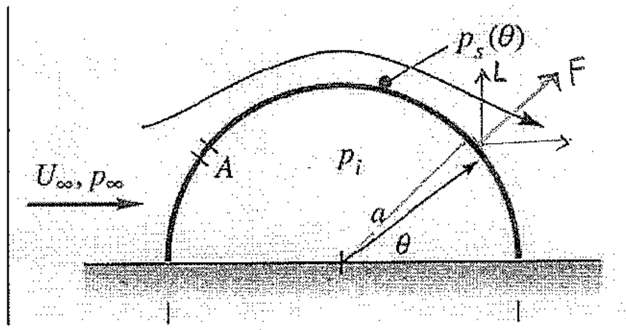
$= U_\infty r\sin\theta + m(\theta_1 + \theta_2)$

(superposition)

The corresponding streamlines for this flow field can be obtained by setting ψ = constant.



#12. A wind with U_∞ and p_∞ flows past a canopy. The canopy is modeled as a half-cylinder of radius a and length L into the paper as shown. The internal pressure is p_i. Using potential flow theory, derive an expression for the upward force on the canopy due to the difference between p_i and p_s.



Solve: On the surface, $u_\theta = -2U_\infty \sin\theta$

By Bernoulli equation

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_s + \frac{1}{2}\rho U_s^2 = p_s + 2U_\infty^2 \sin^2\theta\,\rho$$

$$\Rightarrow p_s = \frac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta) + p_\infty$$

$$dF = p\,dA = -(p_s - p_i)\,a\,d\theta \cdot L$$

$$dL = dF \sin\theta = -(p_s - p_i)\sin\theta\, aL\, d\theta$$

$$\therefore L = \int_0^\pi -aL\left[\frac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta) + p_\infty - p_i\right]\sin\theta\, d\theta$$

$$= aL\left(p_i - p_\infty - \frac{1}{2}\rho U_\infty^2\right)\int_0^\pi \sin\theta\, d\theta - 2\rho U_\infty^2 aL \int_0^\pi \sin^3\theta\, d\theta$$

$$= 2aL\left(p_i - p_\infty - \frac{1}{2}\rho U_\infty^2\right) + 2\rho U_\infty^2 La \cdot \frac{4}{3}$$

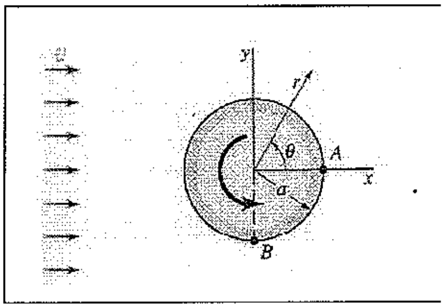
$$= 2La(p_i - p_\infty) + \frac{5}{3}\rho U_\infty^2 La$$

#13. A spinning cylinder is placed in a uniform irrotational and incompressible flow as shown below. For what angular velocity Ω will the stagnation point be located at:

a) point A

b) point B

Which case produces the greatest lifting force?



Solve: Adding a free votex to velocity potential for the flow around a cylinder.

$$\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

So, the tangential velocity V_θ on the surface of cylinder becomes

$$V_{\theta s} = -\frac{\partial\psi}{\partial r}\Big|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a}$$

For stagnation point, $V_\theta = 0$.

$$\Gamma = 4\pi a U\sin\theta$$

At point A: $\theta = 0 \quad \Rightarrow \Gamma = 0$

At point B, $\theta = \frac{3}{2}\pi$, $\Rightarrow \Gamma = -4\pi a U$

$$\Omega = \frac{V_\theta}{a} = \frac{\Gamma}{2\pi a^2} = \frac{-4\pi a U}{2\pi a^2} = -\frac{2U}{a}$$

At point B: $F_y = -\rho U\Gamma = 4\pi a\rho U^2$, the greatest lifting force