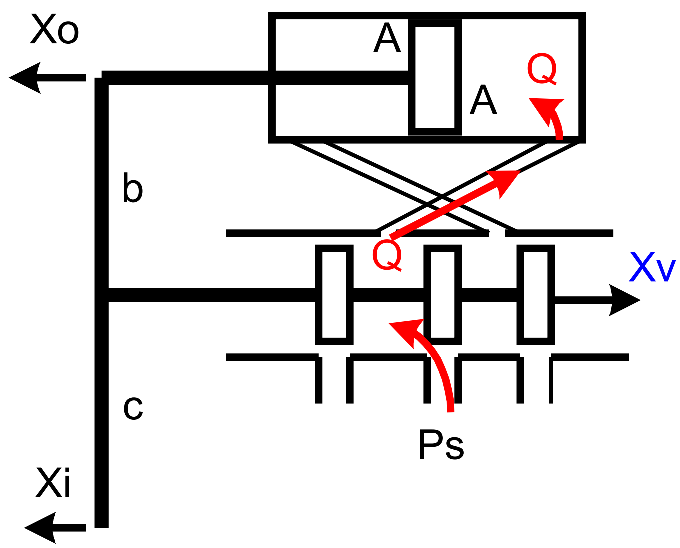
Exam #3 Solution MAE3319 November 15, 2007

Name ______________________________

1. A valve controlled actuator is shown below. x_i is the input and x_o is the output. Assume the fluid is incompressible and the valve is linear. Also assume no force is required to move the massless piston.
   (a) Label the flow variable and x_v and then write the corresponding equations for this system. Determine the transfer function relating x_i to x_o.
   (b) For a step input, how long will it take for the output to reach steady state?



For X_v and Q defined positive as shown, we get the following equations

$$Q = C_v X_v$$

$$Q - A\dot{X}_o = 0$$

Using superposition, the linkage equation is found to be

$$X_v = \frac{-c}{c+b} X_o + \frac{-b}{c+b} X_i$$

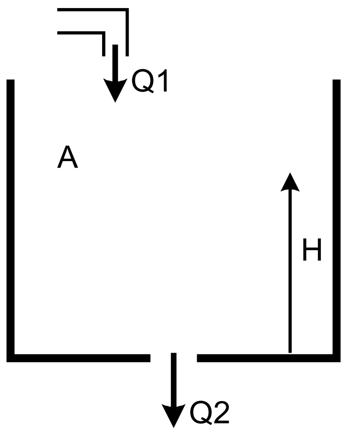
Laplace transforming and then eliminating Q and X_v gives

$$X_o(s) = -\frac{b}{c}\frac{1}{\tau s + 1} X_i(s)$$

$$\tau = \frac{(b+c)A}{cC_v}$$

The time to steady state is 5τ.

2. The water tank shown below has a constant input flow $Q_1 = 0.0376$ m^3/s. The flow through the hole at the bottom requires the orifice equation with $C_d = 0.6$ and hole area $A_o = 0.01$ m^2. Gravity is $g = 9.8$ m/s^2 and the density of the water is $\rho = 1000$ kg/m^3. The area of the tank is $A = 0.50$ m^2.



(a) Derive the differential equation for the height of the water in the tank.

$$Q_1 - Q_2 - A\dot{H} = 0$$

$$Q_2 = C_d A_o \sqrt{\frac{2}{\rho} P} = C_d A_o \sqrt{2gH}$$

Eliminating Q_2 gives

$$A\dot{H} + C_d A_o \sqrt{2gH} = Q_1$$

Substituting for the given parameters gives

$$\dot{H} + 0.05313\sqrt{H} = 0.0752$$

(b) Determine the steady state height, H_{ss}, of the water in the tank.

$$0 + 0.05313\sqrt{H_{ss}} = 0.0752$$

Thus,

$$H_{ss} = 2 \text{ m}$$

(c) Obtain a linear differential equation by linearizing $\sqrt{H}$ for values of H in the neighborhood of H_{ss}.

Linearizing in the neighborhood of $H = 2$ gives

$$\sqrt{H} \approx \sqrt{2} + \frac{1}{2\sqrt{2}}(H - 2) = 0.3536H + 0.707$$

Thus,

$$\dot{H} + 0.05313(0.3536H + 0.707) = 0.0752$$

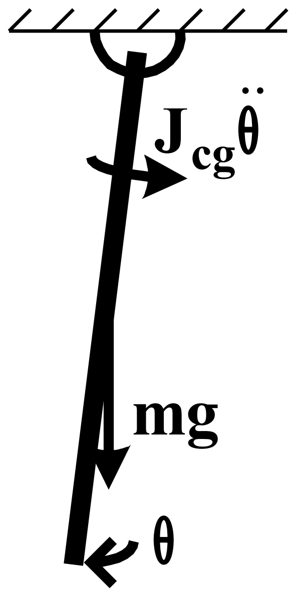
Or,

$$\dot{H} + 0.01879H = 0.03763$$

3. The uniform mass bar shown below swings like a pendulum. The length of the bar is **L** and the mass of the bar is **M**. Denote gravity by **g**. For the bar, $J_{cg} = \frac{ML^2}{12}$.

(a) Neglecting all friction, derive the differential equation for the angular position of the bar if it is released from some non-vertical position.

(b) Using the small angle approximation, estimate the oscillation frequency.



$$\left[m\left(\frac{L}{2}\right)^2 + J_{cg}\right]\ddot{\theta} + mg\frac{L}{2}\sin\theta = 0$$

For small angles, $\sin\theta \approx \theta$

Thus, the differential equation becomes

$$\ddot{\theta} + \omega_n^2\theta = 0$$

The oscillation frequency $\omega_n = \sqrt{\frac{3g}{2L}}$

Useful Equations For Fluids

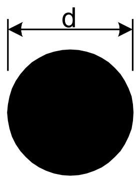
Liquids

(Q m^3/s, P Newtons/m^2 absolute or gauge, d m^2, A & A_v m^2, L m)

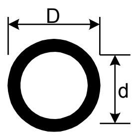
Resistance

$P_u - P_d = RQ$ **where**

$$R = \frac{128\mu L}{\pi d^4}$$ Laminar flow through a circular tube, diameter d



$$R = \frac{128\mu L}{\pi (D-d)^3 (3d-D)}$$ Laminar flow between concentric tubes, diameters d & D



For an orifice or turbulent flow in a tube:

$$P_u - P_d = \frac{\rho}{2C_d^2 A_o^2} Q|Q| \quad \text{or} \quad Q = C_d A_o \sqrt{\frac{2|P_u - P_d|}{\rho}} \operatorname{sign}(P_u - P_d)$$

For a valve, use the $C_d \approx 0.6$ for a sharp edged orifice.

For turbulent flow in a tube use $C_d = \sqrt{\frac{d}{Lf}}$ $\quad f = 0.3164\Re^{-0.25}$

$$\Re = \frac{\rho \bar{u} d}{\mu} = \frac{4\rho Q}{\mu \pi d}$$ which is the Reynolds Number

Inertance

$$P_u - P_d = \frac{\rho L}{A}\dot{Q} \quad \text{or} \quad P_u - P_d = I\dot{Q}$$

Capacitance

$$Q_{in} - Q_{out} - \dot{V} = \frac{V}{\beta}\dot{P} \quad \text{or} \quad Q_{in} - Q_{out} = C\dot{P}$$