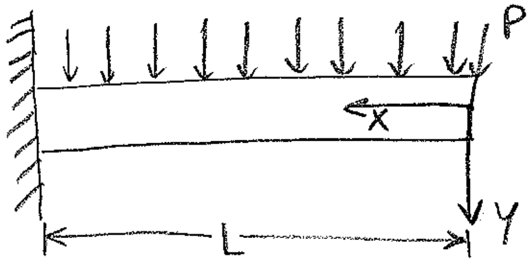
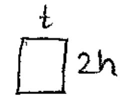
SOLUTION HW # 3

Solu 1





a) Airy stress function

To determine airy stress function for beam subjected to uniformly distributed load P in y-dir
The following conditions must satisfy:

1) σ_y is not function of X, that gives
$$a_3 = a_4 = b_4 = a_5 = b_5 = c_5 = 0$$
2) σ_y is function of y.
3) τ_{xy} is an even function of 'y' which gives
$$c_3 = c_4 = c_5 = e_5 = 0$$

$$e_4 = -(\cancel{a_4}^{0} + 2\cancel{c_4}^{0}) = 0$$
$$e_5 = -(3\cancel{a_5}^{0} + 2\cancel{c_5}^{0}) = 0$$
$$b_5 = -(2d_5 + 3f_5) = 0 \qquad \therefore \quad 2d_5 = -3f_5$$

- Airy stress function

$$\phi = \frac{1}{2} a_2 x^2 + b_2 xy + \frac{1}{2} c_2 y^2 + \frac{1}{2} b_3 x^2 y$$
$$+ \frac{1}{6} d_3 y^3 + \frac{1}{6} d_4 xy^3 + \frac{1}{12} e_4 y^4$$
$$+ \frac{1}{6} d_5 x^2 y^3 + \frac{1}{20} f_5 y^5$$

substituting $e_4 = 0$, and $f_5 = -\frac{2}{3} d_5$

$$\phi = \frac{1}{2} a_2 x^2 + b_2 xy + \frac{1}{2} c_2 y^2 + \frac{1}{2} b_3 x^2 y + \frac{1}{6} d_3 y^3 + \frac{1}{6} d_4 xy^3 + \frac{1}{6} d_5 x^2 y^3 - \frac{1}{30} d_5 y^5$$

b) The beam subject to following boundary conditions

1) $(\tau_{xy})_{y=\pm h} = 0$ at $x = 0$

2) $(\sigma_y)_{y=+h} = 0$ 3) $(\sigma_y)_{y=-h} = -P/t$

4) $\int_{-h}^{h} \sigma_x \, t \, dy = 0 \Big|_{x=0}$ or $\sigma_x = 0$ at $x = 0$

5) $\int_{-h}^{h} \sigma_x \, t \, y \, dy = 0 \Big|_{x=0}$ (bending couple)

6) $\int_{-h}^{h} \tau_{xy} \, t \, dy = -PL \Big|_{x=L}$ (For y equilibrium)

Now, $\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = c_2 + d_3 y + d_5 x^2 y - \frac{2}{3} d_5 y^3$ --- (1)

$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = a_2 + b_3 y + \frac{d_5}{3} y^3$ --- (2)

$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\left(b_2 + b_3 x + \frac{1}{2} d_4 y^2 + d_5 x y^2 \right)$ --- (3)

Applying above boundary conditions

Applying b.C. 4

$$\int_{-h}^{h} \sigma_x \cdot t \cdot dy = 0 \Big|_{x=0}$$

$$\therefore \int_{-h}^{h} \left(c_2 + d_3 y + 0 - \frac{2}{3} d_5 y^3 \right) t \cdot dy = 0$$

$$\therefore \left(c_2 y + d_3 \frac{y^2}{2} - \frac{2}{3} d_5 \frac{y^4}{4} \right)_{-h}^{h} \cdot t = 0$$

$$\therefore c_2 (2h) + 0 + 0 = 0$$

$$\therefore \boxed{c_2 = 0} \quad \text{--- (a)}$$

Applying b.c (2) & (3)

$$\sigma_y = 0 \Big|_{y=+h} \Rightarrow a_2 + b_3 h + \frac{d_5}{3} h^3 = 0 \quad \text{--- (4)}$$

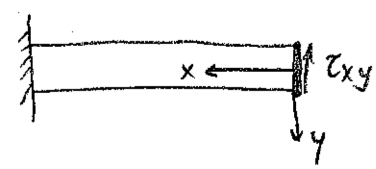
$$\sigma_y = -\frac{p}{t} \Big|_{y=-h} \Rightarrow a_2 - b_3 h - \frac{d_5}{3} h^3 = -\frac{p}{t} \quad \text{--- (5)}$$

From above 2 equations $\Rightarrow \boxed{a_2 = -\frac{p}{2t}}$ --- (b)

$$-\frac{p}{2t} + b_3 h + \frac{d_5}{3} h^3 = 0 \qquad \boxed{\therefore b_3 h + \frac{d_5}{3} h^3 = \frac{p}{2t}} \quad \text{--- (6)}$$

Applying b.c.-1

$$\tau_{xy} = 0 \Big|_{y=\pm h} \quad \text{and } x = 0$$



$$-\left(b_2 + \frac{1}{2} d_4 h^2 \right) = 0$$

$$\boxed{\therefore b_2 = -\frac{1}{2} d_4 h^2} \quad \text{---- (4)}$$

and also $b_3 + d_5 h^2 = 0$

$$\boxed{\therefore b_3 = -d_5 h^2} \quad \text{--- (7)}$$

From equations (6) & (7)

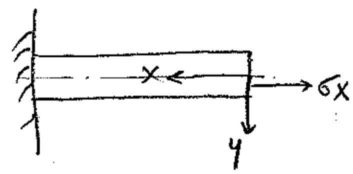
$$-\frac{2}{3} d_5 h^3 = \frac{p}{2t}$$

$$\boxed{d_5 = -\frac{3p}{4th^3}} \quad \text{----(c)}$$

$$\boxed{b_3 = -\frac{3p}{4th}} \quad \text{----(d)}$$

Applying b.c. (5)

$$\int_{-h}^{h} \sigma_x \, t \, y \, dy = 0 \Big|_{x=0}$$



$$\int_{-h}^{h} \left(c_2 + d_3 y + \cancelto{0}{d_5 x^2 y} - \frac{2}{3} d_5 y^3 \right) t \, y \, dy = 0$$

$$\therefore \left(c_2 \frac{y^2}{2} + d_3 \frac{y^3}{3} - \frac{2}{3} d_5 \frac{y^5}{5} \right)_{-h}^{h} t = 0$$

$$\therefore \frac{d_3}{3}(2h^3) - \frac{2}{15} d_5 (2h^5) = 0$$

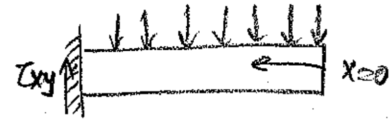
Substituting value of 'd_5' from equ^n (c)

$$\therefore \frac{d_3}{3}(2h^3) - \frac{2}{15}\left(-\frac{3p}{4th^3}\right) 2h^5 = 0$$

$$\boxed{d_3 = -\frac{3p}{10th}} \quad \text{---(e)}$$

Applying b.c. (6)

$$\int_{-h}^{h} \tau_{xy} \, t \, dy = -pL \Big|_{x=L}$$



$$-\int_{-h}^{h}\left(b_2 + b_3 L + \frac{1}{2} d_4 y^2 + d_5 \cdot L \cdot y^2\right) t \, dy = -pL$$

$$\therefore -t\left(b_2 y + b_3 L y + \frac{1}{2} d_4 \frac{y^3}{3} + d_5 \cdot L \cdot \frac{y^3}{3}\right)_{-h}^{h} = -pL$$

$$\left(2b_2 h + 2b_3 L h + \frac{d_4}{3} h^3 + \frac{2 d_5 L}{3} h^3\right) = \frac{pL}{t}$$

Substituting value of b_2, b_3 & b_5 from equ^ns (4, (c) & (d))

$$\therefore \left[-d_4 h^3 + 2Lh\left(\frac{3p}{4th}\right) + \frac{d_4}{3} h^3 + \frac{2}{3} L h^3 \left(-\frac{3p}{4th^3}\right)\right] = \frac{pL}{t}$$

$$\therefore -\frac{2}{3} d_4 h^3 + \frac{3pL}{2t} - \frac{pL}{2t} = \frac{pL}{t}$$

$$\therefore -\frac{2}{3} d_4 h^3 = 0 \qquad \boxed{\therefore d_4 = 0}$$

hence $\boxed{b_2 = 0}$ (f)

(c)

Substituting value of constants from equ^ns a, b, c, d, e, f

$$\sigma_x = 0 - \frac{3p}{10th} y - \frac{3p}{4th^3} x^2 y - \frac{2}{3}\left(-\frac{3p}{4th^3}\right) y^3$$

Now, $I = \frac{1}{12} \times t \times (2h)^3 = \frac{2}{3} t h^3$

$$\therefore \sigma_x = -\frac{ph^2}{5I} y - \frac{p}{2I} x^2 y + \frac{p}{3I} y^3$$

$$= -\frac{p}{10I}\left(5x^2 + 2h^2\right) y + \frac{p}{3I} y^3$$

$$\sigma_y = a_2 + b_3 y + \frac{d_5}{3} y^3$$

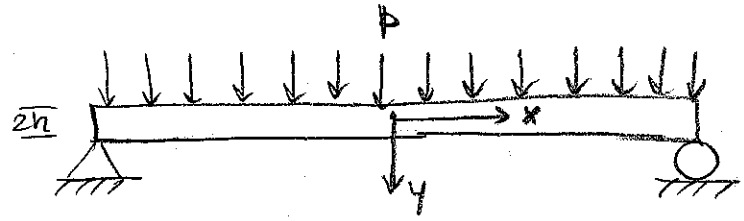
$$= -\frac{p}{2t} + \frac{3p}{4th} y - \frac{3p}{4th^3} \times \frac{y^3}{3}$$

$$= -\frac{ph^3}{3I} + \frac{ph^2}{2I} y - \frac{py^3}{6I}$$

$$= -\frac{p}{6I} \left(2h^3 - 3h^2 y + y^3 \right)$$

$$\tau_{xy} = -\frac{px}{I} (h^2 - y^2)$$

Solu^n 2



(a) To find airy stress function, following conditions must satisfied

1) σ_y is not function x, only function of y.

$\therefore\ a_3 = a_4 = b_4 = a_5 = b_5 = c_5 = 0$

2) τ_{xy} is an even function of 'y' which gives

$c_3 = c_4 = c_5 = e_5 = 0$

3) τ_{xy} is an odd function of 'x', as symmetry about 'y' axis.

$c_2 = d_4 = 0$

4) $e_4 = 0$, $e_5 = 0$ & $d_5 = -3/2\, f_5$

$\therefore$ Airy stress function

$$\phi = \frac{1}{2}a_2x^2 + b_2xy + \frac{1}{2}b_3x^2y + \frac{1}{6}d_3y^3 + \frac{1}{6}d_5x^2y^3 + \frac{1}{20}f_5y^5$$

$$= \frac{1}{2}a_2x^2 + b_2xy + \frac{1}{2}b_3x^2y + \frac{1}{6}d_3y^3 + \frac{1}{6}d_5x^2y^3 - \frac{1}{30}d_5y^5$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = c_2 + d_3y + d_5x^2y - \frac{2}{3}d_5y^3 \quad \text{--- (1)}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = a_2 + b_3y + \frac{d_5}{3}y^3 \quad \text{--- (2)}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\left(b_2 + b_3x + \frac{1}{2}d_4y^2 + d_5xy^2\right) \quad \text{--- (3)}$$

(b) The following boundary conditions to be applied

1) $(\tau_{xy})_{y=\pm h} = 0 \Big|_{x=0}$

2) $(\sigma_y)_{y=+h} = 0 \Big|_{x=0}$ 3) $(\sigma_y)_{y=-h} = -P/t \Big|_{x=0}$

4) $\sigma_x = 0 \Big|_{x=\pm L}$ or $\int_{-h}^{h} \sigma_x t \, dy = 0$

5) Bending couple $\int_{-h}^{h} \sigma_x t y \, dy = 0 \Big|_{x=\pm L}$

6) For y equilibrium $\int_{-h}^{h} \tau_{xy} t \, dy = \pm PL \Big|_{x=\pm L}$

Applying b.c. (1), (2) & (3) to equations (1), (2) & (3)

$$- b_3 - d_5 x y^2 = 0$$

$$a_2 + b_3 h + \frac{d_5}{3} h^3 = 0$$

$$a_2 - b_3 h - \frac{d_5}{3} h^3 = -\frac{P}{t}$$

solving the above (3) equations

$$a_2 = -\frac{P}{2t} \qquad b_3 = \frac{3P}{4th} \qquad d_5 = -\frac{3P}{4th^3} \quad \text{---(a)}$$

Applying b.c. (4)

$$C_2 = 0 \quad \text{---(b)}$$

Applying b.c. (5)

$$\int_{-h}^{h} \left(d_3 y + d_5 L^2 y - \frac{2}{3} d_5 y^3 \right) t y \, dy = 0$$

$$\therefore d_3 = -d_5 \left(L^2 - \frac{2}{5} h^2 \right)$$

$$d_3 = \frac{3P}{4th} \left(\frac{L^2}{h^2} - \frac{2}{5} \right) \quad \text{---(c)}$$

by substituting these values in equ^n 1, 2, & 3

$$I = \frac{2}{3} t h^3$$

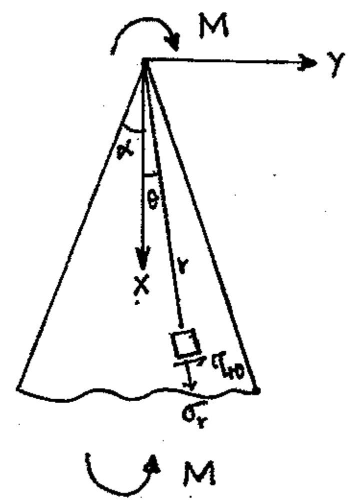
$$\sigma_x = \frac{Py}{2I} \left(L^2 - x^2 \right) + \frac{Py}{I} \left(\frac{y^3}{3} - \frac{h^2}{5} \right)$$

$$\sigma_y = -\frac{p}{2I}\left(\frac{y^3}{3} - h^2 y + \frac{2h^3}{3}\right)$$

$$\tau_{xy} = \frac{-Px}{2I}(h^2 - y^2)$$

Problem 3

Given:



Solution

a) At vertex, $\Sigma M = 0$

$$\Rightarrow 2\int_0^{\alpha} \tau_{r\theta} \cdot r\,d\theta \cdot t \cdot r = +M \cdot t$$

$$\therefore 2\int_0^{\alpha} \tau_{r\theta} \cdot r^2\,d\theta = +M \qquad \text{——— (1)}$$

$\because$ R.H.S. of (1) is constant, L.H.S. also should be constant.

$$\Rightarrow \tau_{r\theta} \propto \frac{1}{r^2}$$

Also,

$$\tau_{r\theta} = \frac{1}{r^2} \cdot \frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta}$$

Considering that $\tau_{r\theta} \propto 1/r^2$, the second term in above expression is zero.

$$\Rightarrow \tau_{r\theta} = \frac{1}{r^2} \cdot \frac{\partial \phi}{\partial \theta}$$

which means Airy stress function ϕ is a function of 'θ' only.

$\therefore \quad \phi = c \cdot f(\theta)$ ——— (2) $\quad c$ = constant.

ϕ must satisfy bi-harmonic equation i.e.

$\nabla^4 \phi = 0$

now,

$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi)$$

$$\nabla^2 \phi = \left[\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right](c \cdot f(\theta))$$

$$= \left[0 + 0 + \frac{c f''(\theta)}{r^2}\right]$$

$$\therefore \nabla^4 \phi = \left[\frac{\partial}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial}{\partial \theta^2}\right]\left(\frac{c \cdot f''(\theta)}{r^2}\right)$$

$$= \left[\frac{6 f''}{r^4} - \frac{2 f''}{r . r^3} + \frac{f^4}{r^4}\right] \cdot c$$

$$\Rightarrow \quad f^4(\theta) + 4 f^2(\theta) = 0$$

Solving this, we get

$$f(\theta) = C_1 + C_2 \cdot \theta + C_3 . \sin 2\theta + C_4 \cdot \cos 2\theta$$

Considering that due to moment acting at vertex, half the beam (+y) would be in compre-ssion and half (-y) in Tension.

$\therefore$ stress function must be an odd function.

$\therefore \quad f(\theta) = -f(-\theta)$

$\therefore \quad C_1 + C_2\theta + C_3 \sin 2\theta + C_4 \cdot \cos 2\theta$

$\qquad = C_1 - C_2\theta - C_3 \sin 2\theta + C_4 \cdot \cos 2\theta$

$\Rightarrow \quad C_1 = C_4 = 0$

$\therefore \quad f(\theta) = C_2\theta + C_3 \sin 2\theta$

$\therefore \quad \phi = c \cdot f(\theta)$

$$= cC_2 \cdot \theta + cC_3 \cdot \sin 2\theta \quad \text{———} \quad (3)$$

$$\tau_{r\theta} = \frac{1}{r^2} \cdot \frac{\partial \phi}{\partial \theta} = \frac{1}{r^2}\left(cC_2 + cC_3 \cdot \cos 2\theta \cdot 2\right)$$

At $\theta = \pm\alpha$, $\tau_{r\theta} = 0$

$$\Rightarrow \quad cC_2 = -2 \cdot cC_3 \cdot \cos 2\alpha \quad \text{———} \quad (4)$$

$$\therefore \quad \tau_{r\theta} = \frac{2 \cdot cC_3}{r^2}\left(\cos 2\theta - \cos 2\alpha\right)$$

Substituting in (1),

$$2\int_0^{\alpha} 2 \cdot \frac{cC_3}{r^2}\left(\cos 2\theta - \cos 2\alpha\right) \cdot r^2 d\theta = +M$$

$$\therefore \quad 4 \cdot cC_3 \left[\frac{\sin 2\theta}{2} - \theta \cdot \cos 2\alpha\right]_0^{\alpha} = +M$$

$$\therefore \quad cC_3 = \frac{+M}{2\left(\sin 2\alpha - 2\alpha \cdot \cos 2\alpha\right)} \quad \text{—} \quad (5)$$

from 4 and 5,

$$CC_2 = \frac{-M.\cos 2\alpha}{(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

$\therefore$ Equation 3 becomes

$$\phi = CC_2 . \theta + CC_3 . \sin 2\theta$$

$$= \frac{-M.\cos 2\alpha.\theta}{(\sin 2\alpha - 2\alpha.\cos 2\alpha)} + \frac{M.\sin 2\theta}{2(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

$$= + \frac{M(\sin 2\theta - 2\theta.\cos 2\alpha)}{2(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

b) Stress Components:

$$\tau_{r\theta} = \frac{2.CC_3}{r^2}(\cos 2\theta - \cos 2\alpha)$$

$$\Rightarrow \tau_{r\theta} = + \frac{M(\cos 2\theta - \cos 2\alpha)}{r^2(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\sigma_r = \frac{1}{r}.\frac{\partial \phi}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 \phi}{\partial \theta^2} = 0 + \frac{1}{r^2}.\frac{+M.(-4\sin 2\theta)}{2(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

$$\Rightarrow \sigma_r = \frac{-2M.\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha.\cos 2\alpha)}$$

c) stress field for semi-infinite plate.

For semi-infinite plate, wedge angle $\alpha = \pi/2$

$$\therefore \quad \sigma_\theta \Big|_{\alpha=\pi/2} = 0$$

$$\sigma_r \Big|_{\alpha=\pi/2} = \frac{-\ 2M \sin 2\theta}{r^2(\sin\pi - \pi.\cos\pi)} = \frac{-2M.\sin 2\theta}{\pi r^2}$$

$$\tau_{r\theta} \Big|_{\alpha=\pi/2} = + \frac{M.(\cos 2\theta - \cos\pi)}{r^2.(\sin\pi - \pi.\cos\pi)} = + \frac{2M.\cos^2\theta}{\pi r^2}$$

Solu^n. 4

$$\sigma = \begin{bmatrix} 120 & 50 & 30 \\ 50 & 80 & 20 \\ 30 & 20 & 10 \end{bmatrix} \text{MPa}$$

Principal stresses can be obtained by solving equation

$$\begin{vmatrix} 120-\sigma_p & 50 & 30 \\ 50 & 80-\sigma_p & 20 \\ 30 & 20 & 10-\sigma_p \end{vmatrix} = 0$$

by Matlab, we get principal stresses

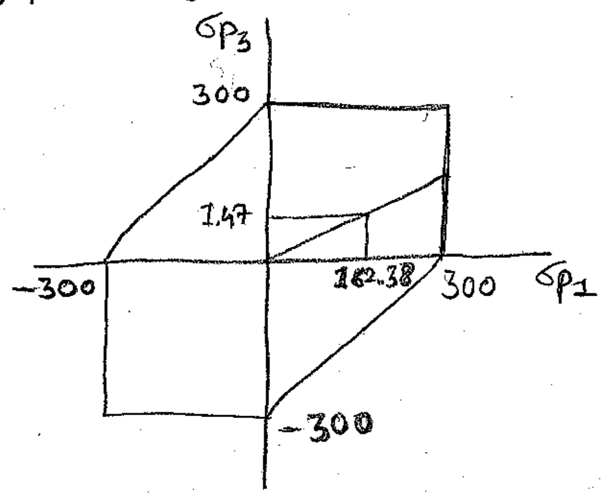
$\sigma_{p1} = 162.38$ MPa $\sigma_{p2} = 46.15$ MPa, $\sigma_{p3} = 1.47$ MPa

$\sigma_{yp} = 300$ MPa

a) According to Maximum Shear Stress theory:

Maximum Shear stress will occur $\frac{(\sigma_{p1} - \sigma_{p3})}{2}$

And as all values are positive, point will lie in 1st quadrant of diagram



$|\sigma_{p1}^{allow}| = 300$

$$S.F. = \frac{\sigma_{p1}^{allow}}{\sigma_{p1}} = \frac{300}{162.38} = 1.847$$

b) Maximum distortion energy theory

According to this theory

$$(\sigma_1^a - \sigma_2^a)^2 + (\sigma_2^a - \sigma_3^a)^2 + (\sigma_3^a - \sigma_1^a)^2 = 2 \cdot (\sigma_{yp})^2$$

Now, $\dfrac{\sigma_1^a}{\sigma_2^a} = \dfrac{\sigma_1}{\sigma_2} = \dfrac{162.38}{46.15} = 3.518$

$\dfrac{\sigma_3^a}{\sigma_2^a} = \dfrac{\sigma_3}{\sigma_2} = \dfrac{1.47}{46.15} = 0.0318$

Substituting in above equation:

$$(3.518\,\sigma_2^a - \sigma_2^a)^2 + (\sigma_2^a - 0.0318\,\sigma_2^a)^2 + (0.0318\,\sigma_2^a - 3.518\,\sigma_2^a)^2 = 2 \cdot (300)^2$$

$$19.43\,(\sigma_2^a)^2 = 2 \cdot (300)^2$$

$$\sigma_2^a = 96.25 \text{ MPa}$$

$$\therefore \quad F.S. = \frac{\sigma_2^a}{\sigma_2} = \frac{96.25}{46.15} = 2.08$$