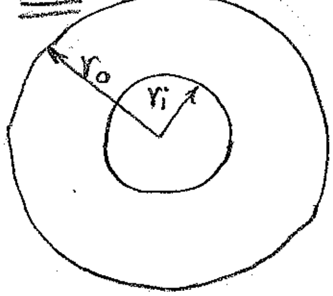
ME 5339 SOLUTION HW#4

Solu^n 4.1 (a)



$$\nabla^4 \phi = 0$$

$$\nabla^2 \phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right] \phi$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right)$$

$$= \sigma_r + \sigma_\theta$$

$$\left(\because \sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \quad \& \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} \right.$$

$$\therefore \sigma_r + \sigma_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial r^2}$$

And ϕ is only function of r, because this is axisymmetric

$$\left. \therefore \sigma_r + \sigma_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} \right)$$

$$\because \nabla^4 \phi = \nabla^2(\nabla^2 \phi) = 0$$

$$= \nabla^2 \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} \right) = 0$$

Hence $\frac{1}{r}\frac{d}{dr} r \frac{d\phi}{dr} = \text{Constant} = a$

$$\frac{d}{dr} r \frac{d\phi}{dr} = r \cdot a$$

$$r \frac{d\phi}{dr} = \frac{r^2 \cdot a}{2} + a_1$$

(Integrating both sides w.r.t. 'r')
(a_1 is integration constant)

$$\frac{d\phi}{dr} = r\frac{a}{2} + \frac{a_1}{r}$$

$$\therefore \phi = \frac{r^2}{4} a + a_1 \ln r + a_3$$

(Again Integrating both sides w.r.t. 'r' and a_3 integration const.)

sustituting $a_2 = \frac{a}{4}$

$$\therefore \phi = a_1 \ln r + a_2 r^2 + a_3 \quad \rightarrow \text{Airy stress function}$$

<u>(b)</u> To determine the constants a_1, a_2 & a_3, use following boundary conditions.

i) At inner surface $r = r_i$ $\quad \sigma_r = -p_i$

$$\text{Now,} \quad \sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \; (\to 0)$$

$$= \frac{1}{r}\left(\frac{a_1}{r} + 2a_2 r\right)$$

$$= \frac{a_1}{r^2} + 2a_2$$

$$\therefore -p_i = \frac{a_1}{r_i^2} + 2a_2 \quad \text{--- (1)}$$

ii) At outer surface $r = r_o$ $\quad \sigma_r = -p_o$

$$\therefore -p_o = \frac{a_1}{r_o^2} + 2a_2 \quad \text{--- (2)}$$

From equ^n (1) & (2)

$$(1) - (2) \Rightarrow -p_i + p_o = \frac{a_1}{r_i^2} + 2a_2 - \frac{a_1}{r_o^2} - 2a_2$$

$$\therefore a_1 = -\frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}$$

And hence $a_2 = \frac{1}{2}\left[\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right]$

(c) Now, we know,

$$\sigma_r = \frac{a_1}{r^2} + 2a_2$$

and $\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = -\frac{a_1}{r^2} + 2a_2$

Substituting values a_1 & a_2 in above eq^n

$$\sigma_r = -\frac{1}{r^2}\left[\frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}\right] + 2\cdot\frac{1}{2}\left[\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right]$$

$$= \left(\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right) - \frac{1}{r^2}\left(\frac{r_i^2 r_o^2 (p_i - p_o)}{r_o^2 - r_i^2}\right)$$

$$\sigma_\theta = \frac{1}{r^2}\left[\frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}\right] + 2\cdot\frac{1}{2}\left[\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right]$$

$$= \left(\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right) + \frac{1}{r^2}\left(\frac{r_i^2 r_o^2 (p_i - p_o)}{r_o^2 - r_i^2}\right)$$

If $K_1 = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$ & $K_2 = \frac{r_i^2 r_o^2 (p_i - p_o)}{r_o^2 - r_i^2}$

Then $\sigma_r = K_1 - \frac{K_2}{r^2}$

& $\sigma_\theta = K_1 + \frac{K_2}{r^2}$

Problem #2:

Part a) Centroid of the section.

Just as before, take the Aluminum as a reference material and substituting $t = 0.04in$ and $a = 12t = 0.48in$,

$$A_{ti}^* = \frac{E_{ti}}{E_{al}} A_{ti} = \frac{16x10^6 \, psi}{10x10^6 \, psi}(2at) = \frac{16x10^6 \, psi}{10x10^6 \, psi}(2(0.48in)(0.04in) = 0.06144in^2$$

$$A_{al1}^* = \frac{E_{al}}{E_{al}} A_{al1} = \frac{10x10^6 \, psi}{10x10^6 \, psi} at = 0.0192in^2$$

$$A_{al2}^* = \frac{E_{al}}{E_{al}} A_{al2} = \frac{10x10^6 \, psi}{10x10^6 \, psi}(a-2t)(t) = 0.016in^2$$

$$A_{al3}^* = \frac{E_{al}}{E_{al}} A_{al3} = \frac{10x10^6 \, psi}{10x10^6 \, psi} at = 0.0192in^2$$

To calculate the centroid,

$$z_c^* = \frac{\sum A_i^* z_i}{\sum A_i^*} = \frac{(0.06144)(a+t/2) + (0.0192)(a/2) + (0.016)(a-t/2) + (0.0192)(a/2)}{0.06144 + 0.0192 + 0.016 + 0.0192}$$

$$z_c^* = \frac{(0.06144)(0.5) + (0.0192)(0.24) + (0.016)(0.46) + (0.0192)(0.24)}{0.11584}$$

$$z_c^* = 0.408in$$

Part b) Equivalent moment of inertia based on Aluminum.

$$I_y = \sum I_{yi}^* = \frac{E_{ti}}{E_{al}}\left(I_0 + Ad^2\right)_{ti} + \left(I_0 + Ad^2\right)_{al}$$

$$I_y = \frac{16x10^6 \, psi}{10x10^6 \, psi}\left(\frac{1}{12}(2a)(t^3) + (2a)(t)(0.408-0.5)^2\right) + \left(\frac{1}{12}(t)(a^3) + at(0.408-0.24)^2\right) +$$
$$+\left(\frac{1}{12}(a-2t)(t^3) + (a-2t)(t)(0.408-0.46)^2\right) + \left(\frac{1}{12}(t)(a^3) + at(0.408-0.24)^2\right)$$

$$I_y = 2.395x10^{-3} in^4$$

$$I_z = \sum I_{zi}^* = \frac{E_{ti}}{E_{al}}\left(I_0 + Ad^2\right)_{ti} + \left(I_0 + Ad^2\right)_{al}$$

$$I_z = \frac{16x10^6\,psi}{10x10^6\,psi}\left(\frac{1}{12}(t)(2a)^3 + (2a)(t)(0)^2\right) + \left(\frac{1}{12}(a)(t^3) + at(0.22)^2\right) + \\ + \left(\frac{1}{12}(t)(a-2t)^3 + (a-2t)(t)(0)^2\right) + \left(\frac{1}{12}(a)(t^3) + at(0.22)^2\right)$$

$$I_z = 6.796x10^{-3}\,in^4$$

And because symmetry,

$$I_{yz} = 0$$

Part c) Max. bending stress and location.

To calculate the constants,

$$k_y^* = \frac{I_y^*}{I_y^* I_z^* - \left(I_{yz}^*\right)^2} = \frac{1}{I_z^*} = 147.1$$

$$k_z^* = \frac{I_z^*}{I_y^* I_z^* - \left(I_{yz}^*\right)^2} = \frac{1}{I_y^*} = 417.5$$

$$k_{yz}^* = \frac{I_{yz}^*}{I_y^* I_z^* - \left(I_{yz}^*\right)^2} = 0$$

Therefore,

$$\sigma_x = (147.1)(50)y + (417.5)(200)z$$

To calculate the neutral axis,

$$\tan\alpha = \frac{I_y M_z - I_{yz} M_y}{I_z M_y - I_{yz} M_z}$$

But $I_{yz} = 0$, $M_y = 200 lb \cdot in$, and $M_z = 50 lb \cdot in$ as a result,

$$\tan\alpha = \frac{I_y M_z}{I_z M_y}$$