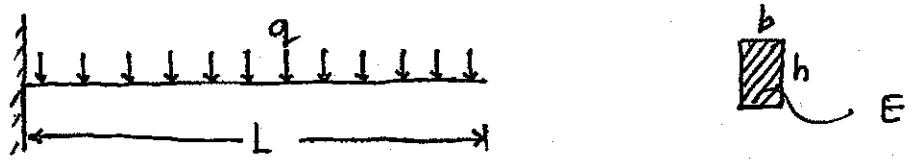
Problem 1

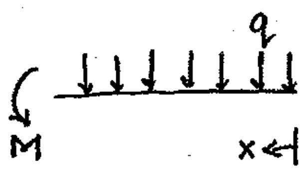
Given :



Solution

a) Bending strain energy

consider section at x from free end:



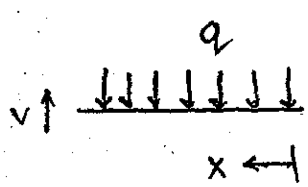
$$\Rightarrow \quad M = (q \cdot x) \cdot \frac{x}{2} = \frac{q \cdot x^2}{2}$$

$$U_b = \int_0^L \frac{M^2}{2EI} \, dx$$

$$= \int_0^L \frac{(qx^2/2)^2}{2EI} \, dx$$

$$= \frac{q^2 L^5}{40EI}$$

b) Strain energy due to shear :



$$\Rightarrow \quad V = q \cdot x$$

$$U_v = \frac{3}{5}\int_0^L \frac{V^2}{GA}\,dx$$

$$= \frac{3}{5}\int_0^L \frac{q^2x^2}{GA}\,dx$$

$$= \frac{q^2L^3}{5GA}$$

c) Ratio U_b/U_v :

$$\frac{U_b}{U_v} = \frac{q^2 L^5}{40EI}\cdot\frac{5GA}{q^2L^3}$$

$$= \frac{L^2GA}{8EI}$$

$$= \frac{3}{4(1+\nu)}\cdot\frac{1}{(h/L)^2} \quad \ldots\ldots \quad G = \frac{E}{2(1+\nu)},\ A = b\times h,\ I = \frac{1}{12}bh^3$$

The ratio h/L is very small for slender beams. As above expression indicates, the ratio of strain energy due to bending to strain energy due to shear is inversely proportional to square of ratio (h/L).

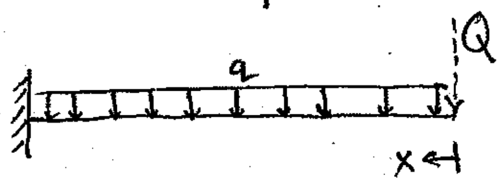
i.e. more the slenderness, more dominant is strain energy due to bending.

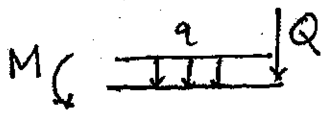
$$\therefore \left.\frac{U_b}{U_v}\right|_{h/L=1/2} = \frac{3}{4(1+\nu)} \cdot \frac{1}{(1/2)^2} = \frac{3}{(1+\nu)}$$

$$\left.\frac{U_b}{U_v}\right|_{h/L=1/3} = \frac{3}{4(1+\nu)} \cdot \frac{1}{(1/3)^2} = \frac{6.75}{(1+\nu)}$$

The values confirm the statement above.

d) Maximum deflection using Castigliano's Th$^{\underline{m}}$
consider point load Q at free end.





$$\Rightarrow \quad M = \frac{q \cdot x^2}{2} + Qx$$

$$\therefore \quad U_b = \int_0^L \frac{M^2}{2EI}\, dx = \frac{1}{2EI}\int_0^L \left(\frac{qx^2}{2} + Qx\right)^2 dx$$

$$\text{deflection } \delta = \frac{\partial U_b}{\partial Q} = \int_0^L \left(\frac{M^2}{2EI}\right)'\Big|_{Q=0} \cdot \frac{\partial M}{\partial Q}\, dx$$

$$= \frac{1}{2EI}\int_0^L 2\cdot\left(\frac{qx^2}{2} + Qx\right)\cdot x\, dx$$

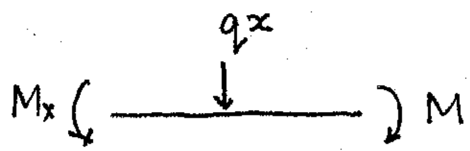
$$= \frac{1}{EI}\int_0^L \left(\frac{qx^3}{2} + Qx^2\right) dx$$

substituting $Q=0$ and solving,

$$\delta = \frac{q \cdot L^4}{8EI}$$

e) Slope at free end:

Consider moment M at free end.



$$\Rightarrow \quad M_x = \frac{qx^2}{2} + M$$

$$U_b = \int_0^L \frac{M_x^2}{2EI}\, dx$$

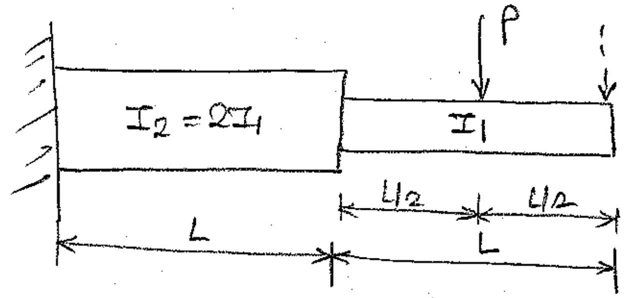
$$\text{slope } \theta = \frac{\partial U_b}{\partial M} = \int_0^L \left(\frac{M_x^2}{2EI}\right)'\Bigg|_{M=0} \frac{\partial M_x}{\partial M}\, dx$$

$$= \int_0^L \frac{2}{2EI}\left(\frac{qx^2}{2} + M\right)_{M=0} \cdot 1 \, dx$$

$$= \frac{qL^3}{6EI}$$

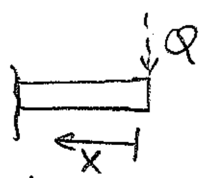
$$\Rightarrow \theta\Big|_{h/L=1/2} = \frac{16q}{E \cdot b} \quad ; \quad \theta\Big|_{h/L=1/3} = \frac{54q}{Eb}$$

Solu^n#2



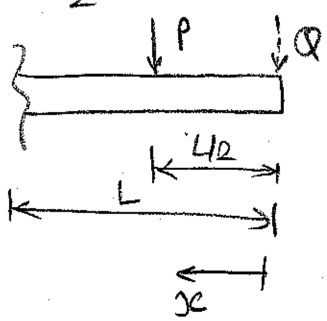
As per castigliano's theorm, to find deflection at free end, consider a dummy force 'Q' at free end as shown in above fig

For $0 \le x \le L/2$



$$M_1 = Q \cdot x \qquad \frac{\partial M_1}{\partial Q} = x$$

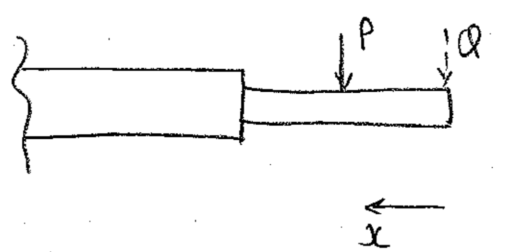
For $\frac{L}{2} \le x \le L$



$$M_2 = Q \cdot x + P\left(x - \frac{L}{2}\right)$$

$$\frac{\partial M_2}{\partial Q} = x$$

For $L \le x \le 2L$



$$M_3 = Q \cdot x + P(x - L/2)$$

$$\frac{\partial M_3}{\partial Q} = x$$

Now, deflection at free end

$$\delta = \frac{\partial U}{\partial Q} = \int_0^{2L} \left(\frac{\partial M}{\partial Q} \cdot \frac{M}{EI}\right)\Bigg|_{Q=0} dx$$

$$= \int_0^{L/2} \frac{\partial M_1}{\partial Q} \frac{M_1}{EI_1}\Bigg|_{Q=0} dx + \int_{L/2}^{L} \frac{\partial M_2}{\partial Q} \cdot \frac{M_2}{EI_1}\Bigg|_{Q=0} dx + \int_L^{2L} \frac{\partial M_3}{\partial Q} \frac{M_3}{EI_2}\Bigg|_{Q=0} dx$$

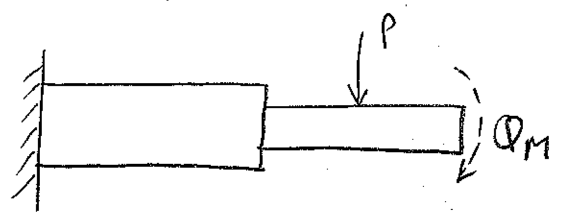
$$\therefore \delta = \frac{\int_0^{L/2} x \cdot 0 \, dx}{EI_1} + \frac{\int_{L/2}^{L} x \cdot P \cdot \left(x - \frac{L}{2}\right) dx}{EI_1} + \frac{\int_{L}^{2L} x \cdot P \cdot \left(x - \frac{L}{2}\right) dx}{EI_2}$$

Substitute $I_2 = 2I_1$

$$\therefore \delta = \frac{1}{EI_1}\left[\frac{5PL^3}{48} + \frac{38PL^3}{48}\right]$$

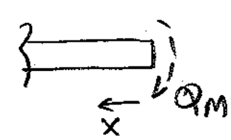
$$\boxed{\therefore \delta = \frac{43PL^3}{48EI_1}}$$ deflection at free end

To find slope at free end, assume a dummy moment Q_M at free end



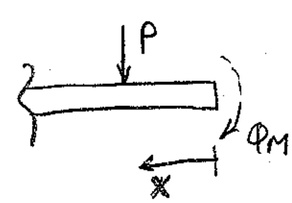
For, $0 \le x \le L/2$

$M_1 = Q_M$ $\therefore \frac{\partial M_1}{\partial Q_M} = 0$



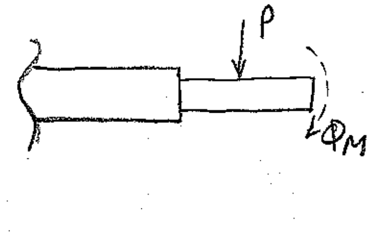
for $L/2 \le x \le L$

$M_2 = Q_M + P\left(x - \frac{L}{2}\right)$ $\frac{\partial M_2}{\partial Q_M} = 1$



for $L \le x \le 2L$

$M_3 = Q_M + P\left(x - \frac{L}{2}\right)$ $\frac{\partial M_3}{\partial Q_M} = 1$



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∴ Slope at free end

$$\theta = \frac{\partial U}{\partial Q_M} = \int_0^{2L} \frac{\partial M}{\partial Q_M}\left(\frac{M}{EI}\right)\Bigg|_{Q_M=0} dx$$

$$= \int_0^{L/2} \left(\frac{\partial M_1}{\partial Q_M}\cdot\frac{M_1}{EI_1}\right)\Bigg|_{Q_M=0} dx + \int_{L/2}^{L} \left(\frac{\partial M_2}{\partial Q_M}\cdot\frac{M_2}{EI_1}\right)\Bigg|_{Q=0} dx + \int_{L}^{2L} \left(\frac{\partial M_3}{\partial Q_M}\cdot\frac{M_3}{EI_2}\right)\Bigg|_{Q=0} dx$$

$$= 0 + \int_{L/2}^{L} 1\cdot\frac{P\left(x-\frac{L}{2}\right)}{EI_1}\,dx + \int_{L}^{2L} 1\cdot\frac{P\left(x-\frac{L}{2}\right)}{E\cdot 2I_1}\,dx$$

$$\boxed{\theta = \frac{5PL^2}{8EI_1}}$$ Slope at free end of beam