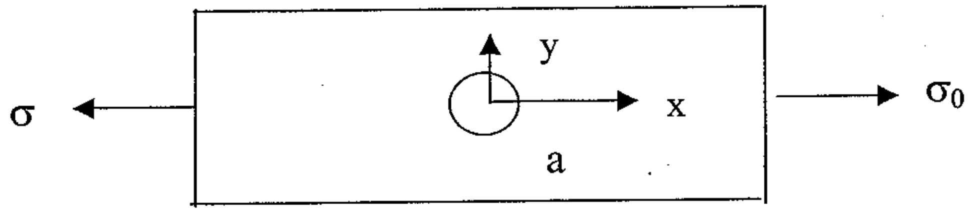
A large, thin plate containing a small circular hole of radius "a" is subjected to a simple tension. The Airy stress fuction can be written as

$\phi = f_1(r) + f_2(r)\cos 2\theta$

Referring to the procedure given in the book, work out the detailed steps and obtain the equations (3.52a) through (3.52c) given in page 125.



$$\phi = f_1(r) + f_2(r)\cos 2\theta$$

$$\nabla^4\phi = \nabla^2(\nabla^2\phi) = 0$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\nabla^2(f_1 + f_2\cos 2\theta) = \nabla^2 f_1 + \nabla^2(f_2\cos 2\theta)$$

$$\nabla^2 f_1 = \frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial^2 f_1}{\partial r}$$

$$\nabla^2 f_2\cos 2\theta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^2}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) f_2\cos 2\theta$$

$$= \frac{\partial^2 f_2}{\partial r^2}\cdot\cos 2\theta + \frac{1}{r}\frac{\partial f_2}{\partial r}\cdot\cos 2\theta - \frac{4}{r^2} f_2\cos 2\theta$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right) f_2\cdot\cos 2\theta$$

$$\nabla^4(f_1 + f_2\cos 2\theta) = 0 \Rightarrow \nabla^4 f_1 = 0, \quad \nabla^4(f_2\cos 2\theta) = 0$$

$$\nabla^4 f_1 = \nabla^2(\nabla^2 f_1) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) f_1 = 0$$

$$\nabla^4 f_2\cos 2\theta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right) f_2\cos 2\theta = 0$$

$$\nabla^4 f_1 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) f_1$$

$$= \frac{1}{r}\Big(\frac{\partial}{\partial r}\ \underbrace{r\frac{\partial}{\partial r}\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Big) f_1}\ = 0$$

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} f_1 = C_1 \rightarrow r\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r\frac{d}{dr} f_1 = 0$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r\frac{d}{dr} f_1 = \frac{C_1}{r} \rightarrow \frac{1}{r}\frac{d}{dr} r\frac{d}{dr} f_1 = C_1 \ln r + C_2$$

$$\frac{d}{dr} r\frac{d}{dr} f_1 = C_1 r\ln r + C_2 r \rightarrow r\frac{d}{dr} f_1 = C_1 \int r\ln r\, dr + C_2\frac{r^2}{2} + C_3$$

$$\int r\ln r\, dr \qquad u = \ln r \quad dv = r\,dr \quad v = \frac{r^2}{2}$$

$$= \int u\,dv = uv - \int v\,du = \frac{r^2}{2}\ln r - \int \frac{r^2}{2}\cdot\frac{1}{r}\,dr = \frac{r^2 \ln r}{2} - \frac{r^2}{4}\cancel{dr} + C_4$$

$$r\frac{d}{dr} f_1 = C_1' \underline{r^2 \ln r} + C_2' r^2 + C_3' \cancel{\ln r} + C_4'$$

$$\frac{df_1}{dr} = C_1' r\ln r + C_2' r + \frac{C_3'}{r}$$

$$f_1 = C_1'' r^2 \ln r + C_2'' r^2 + C_3'' \ln r + C_4''$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)f_2=0 \quad (\text{dropping } \cos 2\theta)$$

$$\text{or} \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)f_2=0$$

The above equation can be solved by two methods:

Method 1: (P.145, Problem 3.28)

The differential equation can be written as

$$r\frac{d}{dr}\Big(\underbrace{\frac{1}{r^3}\frac{d}{dr}\Big\{r^3\frac{d}{dr}\Big[\frac{1}{r^3}\frac{d}{dr}(r^2f_2)\Big]\Big\}}_{\text{is a constant, } C_1}\Big)=0$$

$$\frac{d}{dr}\Big\{r^3\frac{d}{dr}\Big[\frac{1}{r^3}\frac{d}{dr}(r^2f_2)\Big]\Big\}=C_1r^3$$

$$r^3\frac{d}{dr}\Big[\frac{1}{r^3}\frac{d}{dr}(r^2f_2)\Big]=\frac{C_1r^4}{4}+C_2 \rightarrow \frac{d}{dr}\Big[\frac{1}{r^3}\frac{d}{dr}(r^2f_2)\Big]=\frac{C_1r}{4}+\frac{C_2}{r^3}$$

$$\frac{1}{r^3}\frac{d}{dr}(r^2f_2)=\frac{C_1r^2}{8}-\frac{2C_2}{r^2}+C_3 \longrightarrow \frac{d}{dr}r^2f_2=\frac{C_1r^5}{8}-2C_2r+C_3r^3$$

$$r^2f_2=\frac{C_1}{48}r^6-C_2r^2+\frac{C_3}{4}r^4+C_4$$

$$\underline{\underline{f_2=C_5r^2+C_6r^4+\frac{C_7}{r^2}+C_8}} \quad (\text{redefining the coeffici}$$

$$\phi = f_1(x) + f_2(r)\cos 2\theta$$

$$= (C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4) + (C_5 r^2 + C_6 r^4 + \frac{C_7}{r^2} + C_8)\cos 2\theta$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$\frac{\partial\phi}{\partial r} = \frac{\partial f_1}{\partial r} + \frac{\partial f_2}{\partial r}\cos 2\theta$$

$$\frac{\partial f_1}{\partial r} = C_1(\ln r)\cdot 2r + C_1 r^2 \cdot \frac{1}{r} + 2C_2 r + C_3\frac{1}{r} = 2C_1 r\ln r + C_1 r + 2C_2 r + \frac{C_3}{r}$$

$$\frac{\partial f_2}{\partial r} = 2C_5 r + 4C_6 r^3 - \frac{2C_7}{r^3}.$$

$$\frac{\partial^2\phi}{\partial r^2} \qquad \frac{\partial^2 f_1}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f_1}{\partial r}\right) = 2C_1\ln r + 2C_1\cdot r\cdot\frac{1}{r} + C_1 + 2C_2 - \frac{C_3}{r^2}$$

$$\frac{\partial^2 f_2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f_2}{\partial r}\right) = 2C_5 + 12C_6 r^2 + \frac{6C_7}{r^4}$$

$$\frac{\partial\phi}{\partial\theta} = f_2\frac{\partial(\cos 2\theta)}{\partial\theta} = -2f_2\sin 2\theta\,; \quad \frac{\partial^2\phi}{\partial\theta^2} = -4f_2\cos 2\theta$$

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \frac{1}{r}\left[\frac{\partial f_1}{\partial r} + \frac{\partial f_2}{\partial r}\cos 2\theta\right] + \frac{1}{r^2}\left[-4 f_2 \cos 2\theta\right]$$

$$= \frac{1}{r}\left[2C_1 r \ln r + C_1 r + 2C_2 r + \frac{C_3}{r} + \left(2C_5 r + 4C_6 r^3 - \frac{2C_7}{r^3}\right)\cos 2\theta\right]$$

$$+ \frac{1}{r^2}\left[-4\left(C_5 r^2 + C_6 r^4 + \frac{C_7}{r^2} + C_8\right)\cos 2\theta\right]$$

$$= 2C_1 \ln r + C_1 + 2C_2 + \frac{C_3}{r^2} + \left(2C_5 + 4C_6 r^2 - \frac{2C_7}{r^4}\right)\cos 2\theta$$

$$- \left(4C_5 + 4C_6 r^2 + \frac{4C_7}{r^4} + \frac{4C_8}{r^2}\right)\cos 2\theta$$

$$= \left[C_1(2\ln r + 1) + 2C_2 + \frac{C_3}{r^2}\right] - \left(2C_5 + \frac{6C_7}{r^4} + \frac{4C_8}{r^2}\right)\cos 2\theta$$

$$\underline{\sigma_r = \left[C_1(2\ln r + 1) + 2C_2 + \frac{C_3}{r^2}\right] - \left(2C_5 + \frac{6C_7}{r^4} + \frac{4C_8}{r^2}\right)\cos 2\theta}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} + \frac{\partial^2 f_2}{\partial r^2}\cos 2\theta$$

$$= \left(2C_1 \ln r + 3C_1 + 2C_2 - \frac{C_3}{r^2}\right) + \left(2C_5 + 12C_6 r^2 + \frac{6C_7}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\left[-2f_2 \sin 2\theta\right]\right) = 2\sin 2\theta \cdot \frac{\partial}{\partial r}\left(\frac{f_2}{r}\right)$$

$$= 2\sin 2\theta\left[-\frac{f_2}{r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r}\right]$$

$$= 2\sin 2\theta\left[-\frac{1}{r^2}\left(C_5 r^2 + C_6 r^4 + \frac{C_7}{r^2} + C_8\right) + \frac{1}{r}\left(2C_5 r + 4C_6 r^3 - \frac{2C_7}{r^3}\right)\right]$$

$$= \left(2C_5 + 6C_6 r^2 - \frac{6C_7}{r^4} - \frac{2C_8}{r^2}\right)\sin 2\theta$$

Rewriting the stresses

$$\left.\begin{aligned}\sigma_r &= \left(2C_2 + \frac{C_3}{r^2}\right) - \left(2C_5 + \frac{6C_7}{r^4} + \frac{4C_8}{r^2}\right)\cos 2\theta \\ \sigma_\theta &= \left(2C_2 - \frac{C_3}{r^2}\right) + \left(2C_5 + \frac{6C_7}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= \left(2C_5 - \frac{6C_7}{r^4} - \frac{2C_8}{r^2}\right)\sin 2\theta\end{aligned}\right\} \quad (1)$$

B.C. ii) gives

$$r = a \qquad \sigma_r\Big|_{r=a} = \left(2C_2 + \frac{C_3}{a^2}\right) - \left(2C_5 + \frac{6C_7}{a^4} + \frac{4C_8}{a^2}\right)\cos 2\theta = 0$$

$$\tau_{r\theta}\Big|_{r=a} = \left(2C_5 - \frac{6C_7}{a^4} - \frac{2C_8}{a^2}\right)\sin 2\theta = 0$$

$$2C_2 + \frac{C_3}{a^2} = 0 \quad (2) \qquad 2C_5 + \frac{6C_7}{a^4} + \frac{4C_8}{a^2} = 0 \quad (3) \qquad 2C_5 - 6\frac{C_7}{a^4} - \frac{2C_8}{a^2} = 0 \quad (4)$$

B.C. i) gives

$$\sigma_r\Big|_{r\to\infty} = 2C_2 - 2C_5\cos 2\theta = \frac{\sigma_0(1+\cos 2\theta)}{2} \Rightarrow \left.\begin{aligned} C_2 &= \frac{\sigma_0}{4} \\ C_5 &= -\frac{\sigma_0}{4}\end{aligned}\right\} \quad (5)$$

$$③ + ④ \qquad 4C_5 + \frac{2C_8}{a^2} = 0 \qquad C_8 = \frac{a^2}{2}\sigma_0$$

$$③ - ④ \qquad \frac{12C_7}{a^4} + \frac{6C_8}{a^2} = 0 \qquad C_7 = \frac{-a^2}{2}C_8 = -\frac{a^4\sigma_0}{4}$$

$$② \longrightarrow \qquad C_3 = -2a^2C_2 = -\frac{a^2\sigma_0}{2}$$

Boundary Conditions

(i) $r \to \infty$ $\quad \sigma_r = \frac{1}{2}\sigma_0(1+\cos 2\theta) = \sigma_0 \cos^2\theta$

$$\sigma_\theta = \sigma_0 \sin^2\theta$$

$$\tau_{r\theta} = -\tau_0 \sin\theta \cos\theta$$

This condition can be obtained from the coordinate transformation as

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} \sigma_x=\sigma_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_x m^2 \\ \sigma_x n^2 \\ -\sigma_x mn \end{pmatrix} = \begin{pmatrix} \sigma_0 \cos^2\theta \\ \sigma_0 \sin^2\theta \\ -\sigma_0 \sin\theta\cos\theta \end{pmatrix}$$

(ii) $r=a$ $\quad \sigma_r = \tau_{r\theta} = 0$

The B.C. (i) gives

$$\sigma_r\Big|_{r\to\infty} = C_1(\infty+1) + 2C_2 + \frac{C_3}{\infty} - \left(2C_5 + \frac{6C_7}{\infty} + \frac{4C_8}{\infty}\right)\cos 2\theta$$
$$= \text{finite value}$$

This implies $C_1 = 0$

$$\sigma_\theta\Big|_{r\to\infty} = \left(2C_1 \cdot \ln\infty + 3C_1 + 2C_2 - \frac{C_3}{\infty}\right) + \left(2C_5 + 12C_6\infty^2 + \frac{6C_7}{\infty}\right)\cos 2\theta$$
$$= \text{finite value}$$

This gives $C_6 = 0$

The the stresses can be written as

$$\sigma_r = \frac{1}{2}\sigma_0\left[\left(1-\frac{a^2}{r^2}\right)+\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta\right]$$

$$\sigma_\theta = \frac{1}{2}\sigma_0\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau_{r\theta} = -\frac{1}{2}\sigma_0\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

$$\sigma_\theta\Big|_{max} = 3\sigma_0 \quad \text{at } r=a \quad \theta = \pm\frac{\pi}{2}$$

$$\sigma_\theta\Big|_{min} = -\sigma_0 \quad \text{at } r=a \quad \theta = 0 \text{ or } \pm\pi$$